CONFLUENT CASES OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH FOUR SINGULAR POINTS

By

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To Mother,

ever uplifting and inspiring

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INTRODUCTION

This is a discussion of ordinary second order linear differential equations with four singular points and the confluent cases which occur by permitting their singular points to coalesce. Chapter I presents a second order differential equation, formal solutions with proof of convergence, solutions valid for large values of $|z|$, irregular singular points and confluence of singular points. Chapter II develops the general form for a differential equation having four regular singular points and then considers solutions of the confluent equation with singularities at $z = 0,1,\infty, \infty$. Using the normalized form of the confluent equation we find a hypergeometric solution with recurrence relations, nth derivative, and sum and product formulas. In chapter III we derive some of the related equations of mathematical physics (see table of contents) from the general equation of chapter II and present a classification of this type equation when the exponent difference is $\frac{1}{2}$.

Chapter I elaborates on pertinent material in [8, chapter 10][1]. Except for the classification in chapter III, see [2, p. 499], the remaining work is believed to be entirely original.

[1] The numbers in brackets refer to the list of references.

CHAPTER I

Second Order Linear Differential Equations

1.1 Definitions. Let the standard form of an ordinary second order linear differential equation be represented as

$$\frac{d^2u}{dz^2} + p(z)\frac{du}{dz} + q(z)\,u = 0, \tag{1-1}$$

where $p(z)$ and $q(z)$ are assumed to be functions of z analytic [8, p. 83] in some domain D except at a finite number of poles. Any point in D at which $p(z)$ and $q(z)$ are both analytic will be called an ordinary point of the equation; any other points of D will be called singular points of the equation. If there exists a point $z=c$ of D such that, when $p(z)$ and $q(z)$ or both have poles at $z=c$, the poles are of such order that $(z-c)p(z)$ and $(z-c)^2q(z)$ are analytic, then $z=c$ is called a regular singular point for the differential equation. Any poles of $p(z)$ and $q(z)$ that are not of this nature are called irregular singular points.

If $z=c$ is a regular singular point, the equation may be written

$$(z-c)^2\frac{d^2u}{dz^2} + (z-c)P(z-c)\frac{du}{dz} + Q(z-c)\,u = 0\ , \tag{1-2}$$

where $P(z-c)$ and $Q(z-c)$ are analytic at $z=c$. Expanding these functions in a Taylor's Series about $z=c$ we have

$$P(z-c) = p_o + p_1(z-c) + \ldots + p_n(z-c)^n + \ldots = \sum_{n=o}^{\infty} p_n(z-c)^n ,$$

and

$$Q(z-c) = q_o + q_1(z-c) + \ldots + q_n(z-c)^n + \ldots = \sum_{n=o}^{\infty} q_n(z-c)^n,$$

where $p_o, p_1, \ldots, q_o, q_1, \ldots$ are constants. These series converge in the domain D_c formed by a circle of radius r (center c) and its interior, where r is chosen sufficiently small so that c is the only singular point of the equation which is in D_c.

Thus

$$p(z) = \frac{P(z-c)}{z-c} \qquad \text{and} \qquad q(z) = \frac{Q(z-c)}{(z-c)^2}$$

in equation (1-1).

1.2 Formal Solutions of Differential Equations. Let us assume a formal solution of the differential equation to be

$$u = (z-c)^{\alpha} \left[1 + \sum_{n=1}^{\infty} a_n(z-c)^n \right] ,$$

where $\alpha, a_1, a_2, \ldots$ are constants to be determined. Assuming that the term-by-term differentiation and multiplication of the series are valid, we have

$$u' = \alpha(z-c)^{\alpha-1}\left[1 + \sum_{n=1}^{\infty} a_n(z-c)^n\right] + (z-c)^{\alpha}\sum_{n=1}^{\infty} a_n n(z-c)^{n-1}$$

$$= (z-c)^{\alpha-1}\left[\alpha + \sum_{n=1}^{\infty}(\alpha+n)a_n(z-c)^n\right],$$

and

$$u'' = (\alpha-1)(z-c)^{\alpha-2}\left[\alpha + \sum_{n=1}^{\infty}(\alpha+n)a_n(z-c)^n\right]$$

$$+ (z-c)^{\alpha-1}\sum_{n=1}^{\infty} n(\alpha+n)a_n(z-c)^{n-1}$$

$$= (z-c)^{\alpha-2}\left[\alpha(\alpha-1) + \sum_{n=1}^{\infty}(\alpha+n)(\alpha+n-1)a_n(z-c)^n\right].$$

Now substituting these expressions for u,u' and u" into equation (1-2) we obtain

$$(1\text{-}3) \qquad (z-c)^{\alpha}\left[\alpha(\alpha-1) + \sum_{n=1}^{\infty} a_n(\alpha+n)(\alpha+n-1)(z-c)^n\right]$$

$$+ (z-c)^{\alpha} P(z-c)\left[\alpha + \sum_{n=1}^{\infty} a_n(\alpha+n)(z-c)^n\right]$$

$$+ (z-c)^{\alpha} Q(z-c)\left[1 + \sum_{n=1}^{\infty} a_n(z-c)^n\right] = 0.$$

Next substitute the series for P(z-c) and Q(z-c) into equation (1-3):

$$(z-c)^{\alpha}\left[\alpha(\alpha-1) + \sum_{n=1}^{\infty} a_n(\alpha+n)(\alpha+n-1)(z-c)^n\right]$$

$$+ (z-c)^{\alpha}\left[p_o + p_1(z-c) + p_2(z-c)^2 + \ldots + p_n(z-c)^n + \ldots\right]\cdot\left[\alpha + \sum_{n=1}^{\infty} a_n(\alpha+n)(z-c)^n\right]$$

$$+ (z-c)^{\alpha}\left[q_o + q_1(z-c) + q_2(z-c)^2 + \ldots + q_n(z-c)^n + \ldots\right]\cdot\left[1 + \sum_{n=1}^{\infty} a_n(z-c)^n\right] = 0.$$

Now equate to zero the coefficients of the successive powers of $(z-c)$:

$(z-c)^{\alpha}$: $\alpha(\alpha-1) + p_o\alpha + q_o = 0,$

$$\alpha^2 + (p_o-1)\alpha + q_o = 0.$$

$(z-c)^{\alpha+1}$: $a_1\alpha(\alpha+1) + a_1p_o(\alpha+1) + \alpha p_1 + a_1q_o + q_1 = 0,$

$$a_1\left[\alpha^2 + \alpha + p_o(\alpha+1) + q_o\right] + \alpha p_1 + q_1 = 0,$$

$$a_1\left[(\alpha+1)^2-\alpha-1 + p_o(\alpha+1) + q_o\right] + \alpha p_1 + q_1 = 0,$$

(1-4) $$a_1\left[(\alpha+1)^2 + (p_o-1)(\alpha+1) + q_o\right] + \alpha p_1 + q_1 = 0.$$

$$(z-c)^{\alpha+2}: \quad a_2(\alpha+2)(\alpha+1) + p_o a_2(\alpha+2) + p_1 a_1(\alpha+1) + P_2\alpha$$

$$+ q_o a_2 + q_1 a_1 + q_2 = 0,$$

$$a_2\Big[(\alpha+2)(\alpha+1) + p_o(\alpha+2) + q_o\Big]$$

$$+ a_1\Big[p_1(\alpha+1) + q_1\Big] + p_2\alpha + q_2 = 0,$$

$$a_2\Big[(\alpha+2)^2 - \alpha - 2 + p_o(\alpha+2) + q_o\Big]$$

$$+ a_1\Big[p_1(\alpha+1) + q_1\Big] + p_2\alpha + q_2 = 0,$$

$$a_2\Big[(\alpha+2)^2 + (p_o-1)(\alpha+2) + q_o\Big]$$

$$+ a_1\Big[p_1(\alpha+1) + q_1\Big] + p_2\alpha + q_2 = 0.$$

And in general we can write:

$$(z-c)^{\alpha+n}: \quad a_n(\alpha+n)(\alpha+n-1) + p_o a_n(\alpha+n) + \sum_{m=1}^{n-1} p_m a_{n-m}(\alpha+n-m)$$

$$+ p_n\alpha + q_o a_n + \sum_{m=1}^{n-1} q_m a_{n-m} + q_n = 0,$$

$$a_n\Big[(\alpha+n)^2 - \alpha - n + p_o(\alpha+n) + q_o\Big]$$

$$+ \sum_{m=1}^{n-1} a_{n-m}\Big[p_m(\alpha+n-m) + q_m\Big] + p_n\alpha + q_n = 0,$$

$$(1\text{-}5) \quad a_n\left[(\alpha+n)^2 + (p_o-1)(\alpha+n) + q_o\right]$$

$$+\sum_{m=1}^{n-1} a_{n-m}\left[p_m(\alpha+n-m) + q_m\right] + \alpha p_n + q_n = 0.$$

The first of these equations which is obtained by equating to zero the coefficients of the lowest power of (z-c) is called the indicial equation. This equation determines two values of α (which may or may not be distinct). If z=c had been an irregular point, the indicial equation would have been at most of the first degree.

To see this, suppose P(z-c) has a simple pole at z=c and expand it in a Taylor's series.

$$P(z-c) = p_{-1}(z-c)^{-1} + p_o + p_1(z-c) + \ldots + p_n(z-c)^n + \ldots$$

$$= \sum_{n=-1}^{\infty} p_n(z-c)^n , \qquad p_{-1} \neq 0,$$

and as before

$$Q(z-c) = q_o + q_1(z-c) + \ldots + q_n(z-c)^n + \ldots = \sum_{n=o}^{\infty} q_n(z-c)^n .$$

Now multiply according to the following scheme [which is equivalent to substituting the series for P(z-c) and Q(z-c) into equation (1-3)] and obtain the indicial equation:

$$q_o + q_1(z-c) + \ldots \;\Big|\; u = (z-c)^{\alpha}\left[1 + a_1(z-c) + \ldots\right].$$

$$p_{-1} + p_o(z-c) + \ldots \;\Big|\; u' = (z-c)^{\alpha-1}\left[\alpha + (\alpha+1)a_1(z-c) + \ldots\right].$$

$$(z-c)^2 \;\Big|\; u'' = (z-c)^{\alpha-2}\left[\alpha(\alpha-1) + (\alpha+1)\alpha a_1(z-c) + \ldots\right].$$

The indicial equation is

$$(z-c)^{\alpha-1}: \qquad p_{-1}\alpha = 0.$$

But $\alpha \not\equiv 0$, and $p_{-1} \not\equiv 0$ or $z=c$ would not be a simple pole which contradicts our assumption.

Or if we assume that $Q(z-c)$ has a simple pole at $z=c$ and expand it in a Taylor's series:

$$Q(z-c) = q_{-1}(z-c)^{-1} + q_o + q_1(z-c) + \ldots + q_n(z-c)^n + \ldots$$

$$= \sum_{n=-1}^{\infty} q_n(z-c)^n, \qquad q_{-1} \neq 0,$$

and

$$P(z-c) = p_o + p_1(z-c) + p_2(z-c)^2 + \ldots + p_n(z-c)^n + \ldots$$

$$= \sum_{n=0}^{\infty} p_n(z-c)^n .$$

Now multiplying similar to the previously mentioned scheme:

$$\begin{array}{l|l} q_{-1}(z-c)^{-1}+ q_o +\ldots & u = (z-c)^{\alpha}\left[1 + a_1(z-c) +\ldots\right] . \\ p_o(z-c)+ p_1(z-c)^2+\ldots & u' = (z-c)^{\alpha-1}\left[\alpha+(\alpha+1)a_1(z-c)+\ldots\right] . \\ (z-c)^2 & u'' = (z-c)^{\alpha-2}\left[\alpha(\alpha-1) +\ldots\right] . \end{array}$$

The indicial equation is

$$(z-c)^{\alpha-1}: \qquad q_{-1} = 0,$$

which is a contradiction of our assumption.

If both P(z-c) and Q(z-c) have simple poles at z=c, then

$$\begin{array}{l|l} q_{-1}(z-c)^{-1}+ q_o+\ldots & u = (z-c)^{\alpha}\left[1 + a_1(z-c) +\ldots\right] . \\ p_{-1}+p_o(z-c) +\ldots & u' = (z-c)^{\alpha-1}\left[\alpha + (\alpha+1)a_1(z-c)+\ldots\right] . \\ (z-c)^2 & u'' = (z-c)^{\alpha-2}\left[\alpha(\alpha-1)+(\alpha+1)\alpha a_1(z-c)+\ldots\right] . \end{array}$$

The indicial equation is

$$(z-c)^{\alpha-1}: \qquad p_{-1}\alpha + q_{-1} = 0,$$

which is first degree in α.

Hence, if z=c is an irregular singular point, the indicial equation is at most of first degree.

Now the roots of the indicial equation are called the exponents of the differential equation at the point z=c. Returning to our indicial equation which we shall call $F(\alpha)$,

(1-6) $$F(\alpha) = \alpha^2 + (p_o-1)\alpha + q_o = 0,$$

let $\alpha=\lambda_1$ and $\alpha=\lambda_2$ be the roots. Each value for α determines, in order, a set of coefficients $a_1, a_2, \ldots$, thus producing two distinct series solutions, provided that $\lambda_1-\lambda_2$ is not an integer (zero included). It will be shown later that when the difference is a non-zero integer then our proof for convergence of the series fails and if the difference is zero, the two solutions are obviously the same.

1.3 Convergence of Formal Solutions. We now present a proof by induction that these series solutions converge.

Assuming that the exponents λ_1 and λ_2 are not equal, choose them so that $Re(\lambda_1) \geq Re(\lambda_2)$, and let $\lambda_1-\lambda_2 = s$. Now since the exponents are $\alpha = \lambda_1, \lambda_2$, we can write (1-o),

$$F(\alpha) = (\alpha-\lambda_1)(\alpha-\lambda_2), \tag{1-7}$$

then for $\alpha = \lambda_1+n$,

$$\begin{aligned} F(\lambda_1+n) &= (\lambda_1+n-\lambda_1)(\lambda_1+n-\lambda_2) \\ &= n(n+s) \ . \end{aligned}$$

Let

$$P(z-c) = p_o + p_1(z-c) + p_2(z-c)^2 + \ldots + p_n(z-c)^n + \ldots,$$

where

$$p_n = \frac{P^{(n)}(c)}{n!}$$

according to Taylor's expansion. By means of repeated differentiation [4, p. 70]

$$P^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{P(z)\,dz}{z^{n+1}} \quad .$$

If we let the curve C be a circle with center at the origin, $z=re^{i\theta}$, then $dz=rie^{i\theta}\,d\theta$. Let M_1 be the maximum value of $|P(z)|$ on C. Then

$$\left|\frac{n!}{2\pi i}\int_C \frac{P(z)dz}{z^{n+1}}\right| \leq \left|\frac{n!}{2\pi i}\right| \cdot \int_C \frac{|P(z)|\cdot|dz|}{|z^{n+1}|}$$

$$\leq \frac{n!}{2\pi}\int_0^{2\pi} \frac{M_1 \cdot |rie^{i\theta}d\theta|}{|r^{n+1}|}$$

$$\leq \frac{M_1 n!}{2\pi r^n}\int_0^{2\pi} d\theta = \frac{M_1 n!}{2\pi r^n}[\theta]_0^{2\pi} = \frac{M_1 n!}{r^n} \quad .$$

Thus

$$\left|P^{(n)}(z)\right| \leq \frac{M_1 n!}{r^n} \quad ,$$

and

$$|p_n| \leq M_1 r^{-n} \quad .$$

Similarly for q_n:

$$|q_n| \leq M_2 r^{-n} \quad ,$$

where M_2 is the maximum value of $|Q(z)|$ on C.

We next show that

$$|\lambda_1 p_n + q_n| < M_3 r^{-n} \quad .$$

$$|\lambda_1 p_n + q_n| \leq |\lambda_1|\cdot|p_n| + |q_n|$$

$$\leq |\lambda_1|\cdot M_1 r^{-n} + M_2 r^{-n} = r^{-n}\left(|\lambda_1|\cdot M_1 + M_2\right) = r^{-n}M_3,$$

where

$$M_3 = |\lambda_1| M_1 + M_2 .$$

Choose M to be larger than M_1, M_2, and M_3 and also $M \geq 1$. Then

(1-8) $$|p_n| < Mr^{-n}; \quad |q_n| < Mr^{-n}; \quad |\lambda_1 p_n + q_n| < Mr^{-n};$$

for $n = 1,2,3,\ldots$.

Now with the aid of equation (1-6), we can write equation (1-4) as

$$a_1 F(\alpha+1) + \alpha p_1 + q_1 = 0$$

and

$$a_1 = \frac{-(\alpha p_1 + q_1)}{F(\alpha+1)} \quad .$$

Now let $\alpha = \lambda_1$, then

$$|a_1| = \frac{|\lambda_1 p_1 + q_1|}{|F(\lambda_1+1)|} < \frac{Mr^{-1}}{|1+s|} < \frac{M}{r} \quad ,$$

since $|1+s| > 1$. That this is true can be seen by letting $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \gamma + \delta i$. Then $s = (\alpha+\beta i)-(\gamma+\delta i)$ $= (\alpha-\gamma)+i(\beta-\delta)$ and

$$|1+s| = \sqrt{(1+\alpha-\gamma)^2 + (\beta-\delta)^2} \quad .$$

If $\beta-\delta = 0$, or $\beta = \delta$, then $\alpha \neq \gamma$ for we have assumed from the beginning that λ_1 and λ_2 are not equal. By our choice of λ_1 and λ_2, $\alpha > \gamma$ and $1 + \alpha - \gamma > 1$ and $|1+s| > 1$. If $\alpha = \gamma$, then $\beta \neq \delta$ and $\sqrt{1 + (\beta-\delta)^2} > 1$ or $|1+s| > 1$. We have now shown that a_n has a maximum for $n = 1$.

Now we assume that

$$|a_n| < M^n r^{-n} \quad , \qquad n = 1,2,\ldots,m-1.$$

From equations (1-5) and (1-6) we can write

$$a_n F(\alpha+n) + \sum_{m=1}^{n-1} a_{n-m}\left[p_m(\alpha+n-m) + q_m\right] + \alpha p_n + q_n = 0,$$

and letting $\alpha = \lambda_1$,

$$a_n = \frac{-\sum_{m=1}^{n-1} a_{n-m}\left[p_m(\lambda_1+n-m) + q_m\right] - \lambda_1 p_n - q_n}{F(\lambda_1+n)} .$$

Now replace n by m and replace m by t. Then

$$(1\text{-}9) \quad |a_m| = \left| \frac{-\sum_{t=1}^{m-1} a_{m-t}\left[p_t(\lambda_1+m-t) + q_t\right] - \lambda_1 p_m - q_m}{F(\lambda_1+m)} \right|$$

$$\leq \frac{\sum_{t=1}^{m-1} |a_{m-t}| \cdot |p_t\lambda_1+q_t| + |\lambda_1 p_m+q_m| + \sum_{t=1}^{m-1} (m-t)|a_{m-t}| \cdot |p_t|}{m|m+s|}$$

$$< \frac{\sum_{t=1}^{m-1} |a_{m-t}| Mr^{-t} + Mr^{-m} + \sum_{t=1}^{m-1} (m-t)|a_{m-t}| \cdot Mr^{-t}}{m\,|1+sm^{-1}|} .$$

For the first summation:

$$(1\text{-}10) \quad \sum_{t=1}^{m-1} |a_{m-t}| Mr^{-t} = M \sum_{t=1}^{m-1} |a_{m-t}| r^{-t}$$

$$= M\left[|a_{m-1}|r^{-1} + |a_{m-2}|r^{-2} + \ldots + |a_2|r^{-m+2} + |a_1|r^{-m+1}\right]$$

$$< M\left[M^{m-1}r^{-m} + M^{m-2}r^{-m} + \ldots + M^2r^{-m} + Mr^{-m}\right]$$

$$< Mr^{-m}\left[M^{m-1} + M^{m-2} + \ldots + M^2 + M\right]$$

$$< Mr^{-m}(m-1)M^{m-1}$$

$$< (m-1)r^{-m}M^m \ .$$

For the second summation:

$$(1\text{-}11) \qquad \sum_{t=1}^{m-1} (M-t)|a_{m-t}| \cdot Mr^{-t}$$

$$= M\left[(m-1)|a_{m-1}|r^{-1} + (m-2)|a_{m-2}|r^{-2} + \ldots + (1)|a_1|r^{-m+1}\right]$$

$$< M\left[(m-1)M^{m-1}r^{-m} + (m-2)M^{m-2}r^{-m} + \ldots + Mr^{-m}\right]$$

$$< Mr^{-m}\left[(m-1)M^{m-1} + (m-2)M^{m-1} + \ldots + M^{m-1}\right]$$

$$< M^m r^{-m}\left[(m-1) + (m-2) + \ldots + 2 + 1\right]$$

$$< M^m r^{-m} \ \frac{m}{2}\ (m-1) \ .$$

Thus

$$|a_m| < \frac{(m-1)r^{-m}M^m + Mr^{-m} + M^m r^{-m}\ \frac{m}{2}\ (m-1)}{m^2|1+sm^{-1}|}$$

$$< \frac{Mr^{-m}M^m - r^{-m}M^m + Mr^{-m} + M^m r^{-m}\ \frac{m}{2}\ (m-1)}{m^2|1+sm^{-1}|}$$

$$< \frac{mr^{-m}M^m - r^{-m}M^m + M^m r^{-m} + M^m r^{-m}\frac{m}{2}(m-1)}{m^2|1+sm^{-1}|}$$

$$< \frac{r^{-m}M^m(m+1)}{2m|1+sm^{-1}|} \quad .$$

But $|1+sm^{-1}| > 1$, for letting $s = \lambda_1-\lambda_2 = \alpha+\beta i - (\gamma+\delta i)$:

$$|1+\frac{s}{m}| = |\frac{m+s}{m}| = \frac{|m+s|}{m}$$

and

$$|m+s| = \sqrt{(\alpha-\gamma+m)^2 + (\beta-\delta)^2} > m ,$$

since λ_1 and λ_2 are distinct.

Thus

$$\frac{|m+s|}{m} > 1 ,$$

and

$$|a_m| < r^{-m}M^m \frac{(m+1)}{2m} < r^{-m}M^m , \qquad \text{for } m > 1.$$

Therefore, by induction, $|a_n| < M^n r^{-n}$ for all values of n.

If the values of the coefficients corresponding to the exponent λ_2 be $a_1', a_2', \ldots$ we want to show, by a proof similar to the previous induction proof, that

$$|a_n'| < M^n \kappa^n r^{-n} ,$$

where κ is the upper bound of $|1-s|^{-1}$, $|1-\frac{s}{2}|^{-1}$, $|1-\frac{s}{3}|^{-1}$, ... Let us begin by showing that such a bound exists for s not a positive integer.

If $0 < |s| < 1$, then

$$\frac{1}{|1-|s||} \geq \frac{1}{|1-sj^{-1}|}, \qquad j = 1,2,3,\ldots,$$

or

$$|1-\frac{s}{j}| \geq |1-|s||, \qquad j = 1,2,3,\ldots.$$

Proof: Let $s = \alpha+\beta i$, (α,β real); then $0 < |s| = \sqrt{\alpha^2+\beta^2} < 1$.

$$\left|1-\frac{s}{j}\right| = \left|1-\frac{\alpha}{j}-\frac{\beta i}{j}\right| = \sqrt{(1-\frac{\alpha}{j})^2+(\frac{\beta}{j})^2} = \sqrt{1-\frac{2\alpha}{j}+\frac{\alpha^2+\beta^2}{j^2}}$$

$$\geq \sqrt{1-\frac{2\sqrt{\alpha^2+\beta^2}}{j}+\frac{\alpha^2+\beta^2}{j^2}} = \sqrt{\left(1-\frac{\sqrt{\alpha^2+\beta^2}}{j}\right)^2}$$

$$\geq 1-\frac{\sqrt{\alpha^2+\beta^2}}{j} \geq 1-\sqrt{\alpha^2+\beta^2} = 1-|s| = \left|1-|s|\right|.$$

If $|s| > 1$, then

$$\left|1-\frac{s}{j}\right| \geq \left|1-\frac{|s|}{j}\right|.$$

Proof: Let $|s| = t+\theta$, where t is the largest integer in $|s|$ and $0 \leq \theta < 1$.

$$\left|1-\frac{s}{j}\right| = \left|1-\frac{\alpha}{j}-\frac{\beta i}{j}\right| = \sqrt{(1-\frac{\alpha}{j})^2+\frac{\beta^2}{j^2}} = \sqrt{1-\frac{2\alpha}{j}+\frac{\alpha^2+\beta^2}{j^2}}$$

$$\geq \sqrt{1-\frac{2\sqrt{\alpha^2+\beta^2}}{j}+\frac{\alpha^2+\beta^2}{j^2}} = \sqrt{\left(1-\frac{\sqrt{\alpha^2+\beta^2}}{j}\right)^2} = \left|1-\frac{|s|}{j}\right|.$$

We desire $\left|1-\frac{|s|}{j}\right|$ to have a minimum value larger than zero, where $j=1,2,3,\ldots,\ t-n,\ldots,\ t-1,t,t+1,\ldots t+\ell,\ldots$ $(n=1,2,3,\ldots,\ t-1;\ \ell=1,2,3,\ldots)$. Difficulty could arise, however, when $j=t$ and if $\theta = 0$. Then the absolute value would be zero.

First we show

$$\left|1-\frac{t+\theta}{t-n}\right| \geq \frac{\theta}{t} \quad \text{and} \quad \left|1-\frac{t+\theta}{t+\ell}\right| \geq \frac{1-\theta}{t+1}$$

analytically, where $n=0,1,2,\ldots,\ t-1;\ \ell=1,2,3,\ldots;\ 0 < \theta < 1$.

$$\left|1-\frac{t+\theta}{t-n}\right| \geq \frac{\theta}{t}, \qquad n=0,1,2,\ldots,\ t-1;\ 0 < \theta < 1.$$

$$\left|\frac{t-n-t-\theta}{t-n}\right| \geq \frac{\theta}{t},$$

$$\frac{n+\theta}{t-n} \geq \frac{\theta}{t},$$

$$nt + \theta t \geq \theta t - n\theta ,$$

$$nt \geq -n\theta, \text{ true for } n=0.$$

If $n\neq 0$, then $t > -\theta$.

$$\left|1-\frac{t+\theta}{t+\ell}\right| \geq \frac{1-\theta}{t+1}, \qquad \ell=1,2,3,\ldots;\ 0 < \theta < 1.$$

$$\left|\frac{t+\ell-t-\theta}{t+\ell}\right| \geq \frac{1-\theta}{t+1},$$

$$\frac{\ell-\theta}{t+\ell} \geq \frac{1-\theta}{t+1},$$

$$(t+1)(\ell-\theta) \geq (1-\theta)(t+\ell),$$

$$t\ell-\theta t+\ell-\theta \geq t+\ell-\theta t-\theta\ell ,$$

$$t\ell+\theta\ell \geq t+\theta ,$$

$$\ell(t+\theta) \geq t+\theta ,$$

$$\ell \geq 1, \text{ which is true.}$$

Therefore,

$$(1\text{-}12) \qquad \left|1-\frac{t+\theta}{t-n}\right| \geq \frac{\theta}{t} \quad \text{and} \quad \left|1-\frac{t+\theta}{t+\ell}\right| \geq \frac{1-\theta}{t+1} ,$$

where $n=0,1,2,\ldots, t-1$; $\ell=1,2,3,\ldots$; $0 < \theta < 1$.

Inserting $|s|$ for $t+\theta$ and j for $t-n$ and $t+\ell$ in (1-12) we see that, for $|s| > 1$ and $0 < \theta < 1$, either

$$\left|1-\frac{|s|}{j}\right| \geq \frac{\theta}{t} \quad \text{or} \quad \left|1-\frac{|s|}{j}\right| \geq \frac{1-\theta}{t+1} ,$$

depending on the values of θ and t.

Now consider the case where $\theta=0$ ($|s|=t$) and $j=t = \sqrt{\alpha^2+\beta^2}$, $\beta\neq 0$. We wish to find a δ such that if $s = \alpha+\beta i$, then

$$\left|1-\frac{s}{j}\right| \geq \left|1-\frac{|s|+\delta}{j}\right| .$$

$$\left|1-\frac{\alpha+\beta i}{\sqrt{\alpha^2+\beta^2}}\right| \geq \left|1-\frac{\sqrt{\alpha^2+\beta^2}+\delta}{\sqrt{\alpha^2+\beta^2}}\right| ,$$

$$|\sqrt{\alpha^2+\beta^2} - \alpha - \beta i| \geq |\delta| ,$$

$$\left(\sqrt{\alpha^2+\beta^2} - \alpha\right)^2 + \beta^2 \geq \delta^2 ,$$

$$\alpha^2+\beta^2 - 2\alpha\sqrt{\alpha^2+\beta^2} + \alpha^2+\beta^2 \geq \delta^2 ,$$

$$2t^2-2\alpha t \geq \delta^2 ,$$

$$2t(t-\alpha) \geq \delta^2 .$$

However, since $|s|=t > 1$ and $t = \sqrt{\alpha^2+\beta^2} > \alpha$, if we have given any value of s ($|s| > 1$, $\beta\neq 0$), then a value of δ can be determined such that $0 < \delta < 1$. Referring to (1-12) with θ replaced by δ, we have for $|s| > 1$, $0 < \delta < 1$, that

$$\left|1-\frac{t+\delta}{t-n}\right| \geq \frac{\delta}{t} \quad \text{and} \quad \left|1-\frac{t+\delta}{t+\ell}\right| \geq \frac{1-\delta}{t+1},$$

where $n=0,1,2\ldots,\ t-1;\ \ell=1,2,3,\ldots$.

Summarizing our results we have:

If $0 < |s| < 1$, then

$$\left|1-\frac{s}{j}\right| \geq 1-|s|.$$

For $|s| > 1$, $|s| \neq$ an integer,

(a) if $0 < \theta < 1$, then one of the following

$$\left|1-\frac{s}{j}\right| \geq \left|1-\frac{|s|}{j}\right| = \left|1-\frac{t+\theta}{j}\right| \geq \begin{cases} \frac{\theta}{t}, \\ \frac{1-\theta}{t+1}, \end{cases}$$

(b) if $\theta = 0$, then one of the following

$$\left|1-\frac{s}{j}\right| \geq \left|1-\frac{|s|+\delta}{j}\right| = \left|1-\frac{t+\delta}{j}\right| \geq \begin{cases} \frac{\delta}{t}, \\ \frac{1-\delta}{t+1}, \end{cases}$$

where $j=1,2,3,\ldots$; $|s|=t+\theta$ (where t is the largest integer in $|s|$ and $0 \leq \theta < 1$); δ chosen such that $0 < \delta < 1$ and $\delta^2 \leq 2t(t-\alpha)$.

We have thus proved the theorem that given any s (s not an integer) an upper bound κ of the sequence

$$\left|1-\frac{s}{j}\right|^{-1}, \qquad j=1,2,3,\ldots,$$

is the maximum of

$$\frac{1}{1-|s|};\ \frac{t}{\theta},\ (\theta\neq 0);\ \frac{t+1}{1-\theta},\ (\theta\neq 0);\ \frac{t}{\delta};\ \frac{t+1}{1-\delta};$$

where δ is chosen such that $\delta^2 \leq 2t(t-\alpha)$ and $0 < \delta < 1$.

Now we present an inductive proof that

$$|a_n'| < M^n \kappa^n r^{-n} .$$

Again using equation (1-6) with α now replaced by $\lambda_2 + n$, we have

$$F(\lambda_2+n) = (\lambda_2+n-\lambda_1)(\lambda_2+n-\lambda_2) = n(n-s) . \tag{1-13}$$

If λ_1 is replaced by λ_2, the inequalities of (1-8) will still be true. However, they will also be found valid if M is chosen larger than $\frac{1}{\kappa}$ as well as M_1, M_2, M_3 and $M \geq 1$. Let this be done. Thus,

$$|p_n| < Mr^{-n}; \quad |q_n| < Mr^{-n}; \quad |\lambda_2 p_n + q_n| < Mr^{-n} .$$

Also, since $M > \frac{1}{\kappa}$ and $M \geq 1$, we have the additional inequalities which will be utilized later in the proof,

$$M\kappa > 1,$$
$$(M\kappa)^{m-1} > 1, \qquad \text{where } m=1,2,3,\ldots .$$

When $\alpha = \lambda_2$ and n=1 in equation (1-4), we can write

$$a_1' = \frac{-\lambda_2 p_1 - q_1}{(\lambda_2+1)^2 + (p_o-1)(\lambda_2+1) + q_o} ,$$

and with equation (1-6) this becomes

$$a_1' = \frac{-\lambda_2 p_1 - q_1}{F(\lambda_2+1)} .$$

Taking the absolute value of each side and using (1-13) with n=1, we obtain the inequality

$$|a_1'| = \frac{|\lambda_2 p_1 + q_1|}{|F(\lambda_2+1)|} < \frac{Mr^{-1}}{|1-s|} .$$

Next, assume

$$|a_n'| < M^n \kappa^n r^{-n} , \qquad n=1,2,3,\ldots, m-1,$$

and show that

$$|a_m'| < M^m \kappa^m r^{-m} .$$

Using (1-9) with λ_2 and a_m' replacing λ_1 and a_m respectively,

$$|a_m'| = \left| \frac{-\sum_{t=1}^{m-1} a_{m-t}' \left[p_t(\lambda_2+m-t) + q_t \right] - \lambda_2 p_m - q_m}{F(\lambda_2+m)} \right|$$

$$\leq \frac{\sum_{t=1}^{m-1} |a_{m-t}'| \cdot |p_t\lambda_2+q_t| + |\lambda_2 p_m + q_m| + \sum_{t=1}^{m-1} (m-t) \cdot |a_{m-t}'| \cdot |p_t|}{m|m-s|}$$

$$< \frac{\sum_{t=1}^{m-1} |a_{m-t}'| \cdot Mr^{-t} + Mr^{-m} + \sum_{t=1}^{m-1} (m-t) \cdot |a_{m-t}'| \cdot Mr^{-t}}{m^2 |1-sm^{-1}|} .$$

Similar to (1-10) and (1-11) with $|a'_{m-t}| < (M\kappa)^m r^{-m}$, $M_\kappa > 1$,

$$\sum_{t=1}^{m-1} |a'_{m-t}| \cdot Mr^{-t} < (m-1)(M\kappa)^{m-1} Mr^{-m}$$

$$< (m-1)r^{-m}M^m\kappa^{m-1} ,$$

and

$$\sum_{t=1}^{m-1} (m-t)|a'_{m-t}| \cdot Mr^{-t} < \frac{m}{2} (m-1)(M\kappa)^{m-1}Mr^{-m}$$

$$< \frac{m}{2} (m-1)r^{-m}M^m\kappa^{m-1} .$$

Thus,

$$|a'_m| < \frac{(m-1)r^{-m}M^m\kappa^{m-1} + Mr^{-m} + M^m r^{-m} \frac{m}{2} (m-1) \kappa^{m-1}}{m^2|1-sm^{-1}|}$$

$$< \frac{mr^{-m}M^m\kappa^{m-1} - r^{-m}M^m\kappa^{m-1} + Mr^{-m} + M^m r^{-m} \frac{m}{2} (m-1) \kappa^{m-1}}{m^2|1-sm^{-1}|} .$$

Now since $(M\kappa)^{m-1} > 1$,

$$Mr^{-m} - M^m r^{-m}\kappa^{m-1} = Mr^{-m}(1-M^{m-1}\kappa^{m-1})$$

$$= Mr^{-m}\left[1-(M\kappa)^{m-1}\right]$$

is negative. Thus,

$$|a'_m| < \frac{mr^{-m}M^m\kappa^{m-1} + r^{-m}M^m \frac{m}{2} (m-1) \kappa^{m-1}}{m^2|1-sm^{-1}|}$$

$$< \frac{r^{-m}M^m\kappa^{m-1}(m+1)}{2m|1-sm^{-1}|}$$

$$< r^{-m}M^m\kappa^m \frac{(m+1)}{2m} < r^{-m}M^m\kappa^m, \qquad \text{for } m > 1.$$

Therefore,

$$|a_n'| < M^n r^{-n}\kappa^n, \qquad \text{for all values of } n.$$

The radius of convergence for the power series

$\sum_{n=1}^{\infty} M^n r^{-n}(z-c)^n$ is $\frac{r}{M}$ [4, p. 80], for the

$$\lim_{n\to\infty} \left| \frac{M^n r^{-n}}{M^{n+1} r^{-n-1}} \right| = \lim_{n\to\infty} \frac{r}{M} = \frac{r}{M} .$$

That is, the series is absolutely convergent within the circle $|z-c| = \frac{r}{M}$. Since $|a_n| < M^n r^{-n}$, the series

$\sum_{n=1}^{\infty} a_n(z-c)^n$ also must converge within the circle $|z-c|$ $= \frac{r}{M}$ and is, therefore, uniformly convergent in the region $|z-c| < \frac{r}{M}$ [4, p. 95].

Similarly $\sum_{n=1}^{\infty} a_n'(z-c)^n$ converges uniformly within the region $|z-c| < \frac{r}{M\kappa}$.

We have thus obtained two formal solutions

$$u_1(z) = (z-c)^{\lambda_1}\left[1 + \sum_{n=1}^{\infty} a_n(z-c)^n\right]$$

and

$$u_2(z) = (z-c)^{\lambda_2}\left[1 + \sum_{n=1}^{\infty} a_n'(z-c)^n\right] ,$$

which are uniformly convergent series of analytic functions when $|z-c| < rM^{-1}$ and $|z-c| < rM^{-1}\kappa^{-1}$, respectively, provided that arg $(z-c)$ is restricted in such a manner that the series are single-valued. Consequently, the formal substitution of these series into the differential equation is justified for $\lambda_1-\lambda_2$ not a positive integer. These solutions are valid in the vicinity of a regular singular point.

1.4 Second Solution in the Case where the Difference of the Exponents is an Integer. We now derive a second solution in the case where the difference of the exponents is an integer. When $\lambda_1-\lambda_2 = s$ is a positive integer or zero, the solution $u_2(z)$ may break down or coincide with $u_1(z)$.

Try the change of variable $u = u_1(z)\cdot\phi$ and substitute this into

$$(z-c)^2 u'' + (z-c)P(z-c)\, u' + Q(z-c)\, u = 0$$

to determine the equation for ϕ.

$$u = u_1 \cdot \phi,$$
$$u' = u_1 \cdot \phi' + u_1' \phi,$$
$$u'' = u_1 \cdot \phi'' + 2u_1' \phi' + u_1'' \phi.$$

Substituting in we have

$$(z-c)^2\left[u_1\phi'' + 2u_1'\phi' + u_1''\phi\right] + (z-c)P(z-c)(u_1\phi' + u_1'\phi)$$

$$+ Q(z-c)\, u_1 \cdot \phi = 0,$$

$$(z-c)^2 u_1\phi'' + \left[2u_1'(z-c)^2 + (z-c)P(z-c)\cdot u_1\right] \phi'$$

$$+ \left[(z-c)^2\, u_1'' + (z-c)P(z-c)\, u_1' + Q(z-c)\, u_1\right] \phi = 0.$$

Since $u_1(z)$ is a solution

$$(z-c)^2\, u''_1 + (z-c)P(z-c)\, u_1' + Q(z-c)\, u_1 = 0,$$

and

$$(z-c)^2\, u_1\phi'' + \left[2u_1'(z-c)^2 + (z-c)P(z-c)\cdot u_1\right] \phi' = 0.$$

Dividing by u_1 we have

$$(z-c)^2\, \phi'' + \left[2\,\frac{u_1'}{u_1}\,(z-c)^2 + (z-c)P(z-c)\right] \phi' = 0.$$

To find a general solution let $\phi' = Y$.

$$(z-c)^2\, Y' + \left[2\,\frac{u_1'}{u_1}\,(z-c)^2 + (z-c)P(z-c)\right] Y = 0,$$

$$(z-c)^2\, \frac{dY}{dz} + \left[2\,\frac{u_1'}{u_1}\,(z-c)^2 + (z-c)P(z-c)\right] Y = 0.$$

Separating variables and integrating,

$$\int \frac{dY}{Y} + \int \left[2\frac{u_1'}{u_1} + \frac{P(z-c)}{(z-c)} \right] dz = \ln B,$$

$$\ln Y = \ln B - 2\ln u_1 - \int \frac{P(z-c)}{(z-c)}\, dz ,$$

$$Y = B\, u_1^{-2}\, e^{-\int \frac{P(z-c)}{(z-c)}\, dz} .$$

Now since $Y = \phi'$,

$$d\phi = \left[B\, u_1^{-2}\, e^{-\int \frac{P(z-c)}{(z-c)}\, dz} \right] dz ,$$

where $P(z-c) = p_o + p_1(z-c) + p_2(z-c)^2 + \ldots + p_n(z-c)^n + \ldots$.

$$\int \frac{P(z-c)}{(z-c)}\, dz = \int \left[p_o(z-c)^{-1} + p_1 + p_2(z-c) + \ldots \right] dz$$

$$= p_o \ln(z-c) + p_1(z-c) + p_2 \frac{(z-c)^2}{2} + \ldots .$$

Hence,

$$\phi = A + B \int u_1^{-2}\, e^{\left[-p_o \ln(z-c) - p_1(z-c) - \ldots - p_n(z-c)^n - \ldots\right]} dz$$

$$= A + B \int u_1^{-2}(z-c)^{-p_o}\, e^{\left[-p_1(z-c) - \ldots - p_n(z-c)^n - \ldots\right]} dz$$

$$= A + B \int (z-c)^{-p_o}\, u_1^{-2}\, e^{-\sum_{n=1}^{\infty} p_n \frac{(z-c)^n}{n}} dz .$$

But since

$$u_1(z) = (z-c)^{\lambda_1}\left[1 + \sum_{n=1}^{\infty} a_n(z-c)^n\right],$$

$$\phi = A + B\int (z-c)^{-p_o-2\lambda_1}\, g(z)dz,$$

where $g(z) = \left[1 + \sum_{n=1}^{\infty} a_n(z-c)^n\right]^{-2} \cdot e^{-\sum_{n=1}^{\infty} p_n \frac{(z-c)^n}{n}}$.

A and B are arbitrary constants and g(z) is analytic throughout the interior of any circle C whose center is z=c, which does not contain any singularities of P(z-c) or of $(z-c)^{-\lambda_1}\, u_1(z)$, nor any zeros of the latter. $g(c) = 1$.

Since g(z) is analytic in the interior of C, it can be expanded in a Taylor's series about its center z=c. Let

$$g(z) = 1 + \sum_{n=1}^{\infty} g_n(z-c)^n .$$

We already have $\alpha^2 + (p_o-1)\,\alpha + q_o = 0$ (1-6) with roots $\alpha = \lambda_1, \lambda_2$ and $\lambda_1-\lambda_2 = s$. The sum of the roots $\lambda_1+\lambda_2 = 1-p_o$, so we can write

$$-p_o-2\lambda_1 = \lambda_1+\lambda_2-1-2\lambda_1 = -\lambda_1+\lambda_2-1 = -s-1.$$

Then

$$(1\text{-}14) \qquad \phi = A + B\int \left[1 + \sum_{n=1}^{\infty} g_n(z-c)^n\right](z-c)^{-s-1}\, dz$$

$$= A + B\Big[\int (z-c)^{-s-1}dz + \int \sum_{n=1}^{s-1} g_n(z-c)^{n-s-1}dz$$

$$+ \int g_s(z-c)^{-1}dz + \int \sum_{n=s+1}^{\infty} g_n(z-c)^{n-s-1}dz\Big]$$

$$= A + B\Big[\frac{(z-c)^{-s}}{-s} - \sum_{n=1}^{s-1} \frac{g_n}{s-n}(z-c)^{n-s} + g_s \ln(z-c)$$

$$+ \sum_{n=s+1}^{\infty} \frac{g_n}{n-s}(z-c)^{n-s}\Big] .$$

The general solution, analytic in C except at z=c, is

$$u = u_1(z)\cdot\phi = A\, u_1(z) + B\Big[u_1(z)\cdot g_s \cdot \ln(z-c) + \bar{u}(z)\Big] ,$$

where

$$\bar{u}(z) = u_1(z)\Big[-\frac{1}{s}(z-c)^{-s} - \sum_{n=1}^{s-1} \frac{g_n}{s-n}(z-c)^{n-s}$$

$$+ \sum_{n=s+1}^{\infty} \frac{g_n}{n-s}(z-c)^{n-s}\Big]$$

and

$$u_1(z) = (z-c)^{\lambda_1}\Big[1 + \sum_{n=1}^{\infty} a_n(z-c)^n\Big] .$$

Rewriting, we have

$$\bar{u}(z) = (z-c)^{\lambda_2+s}\left[1 + \sum_{n=1}^{\infty} a_n(z-c)^n\right]\cdot\left[-\frac{1}{s}(z-c)^{-s} - \sum_{n=1}^{s-1}\frac{g_n}{s-n}(z-c)^{n-s}\right.$$

$$\left. + \sum_{n=s+1}^{\infty}\frac{g_n}{n-s}(z-c)^{n-s}\right]$$

$$= (z-c)^{\lambda_2}\left[1 + \sum_{n=1}^{\infty} a_n(z-c)^n\right]\left[-\frac{1}{s} - \sum_{n=1}^{s-1}\frac{g_n}{s-n}(z-c)^n\right.$$

$$\left. + \sum_{n=s+1}^{\infty}\frac{g_n}{n-s}(z-c)^n\right]$$

$$= (z-c)^{\lambda_2}\left[-\frac{1}{s} + \sum_{n=1}^{\infty} h_n(z-c)^n\right],$$

where h_n are constants:

$$h_1 = -\left(\frac{g_1}{s-1} + \frac{a_1}{s}\right)$$

and

$$\sum_{n=1}^{\infty} h_n(z-c)^n = -\frac{1}{s}\sum_{n=1}^{\infty} a_n(z-c)^n + \left[1 + \sum_{n=1}^{\infty} a_n(z-c)^n\right]$$

$$\cdot\left[\sum_{n=1}^{s-1}\frac{g_n}{n-s}(z-c)^n + \sum_{n=s+1}^{\infty}\frac{g_n}{n-s}(z-c)^n\right].$$

When s = 0, we can write (1-14) in the form

$$\phi = A + B \int \left[1 + \sum_{n=1}^{\infty} g_n (z-c)^n \right] (z-c)^{-1} \, dz$$

$$= A + B \left[\int (z-c)^{-1} dz + \int \sum_{n=1}^{\infty} g_n (z-c)^n dz \right]$$

$$= A + B \left[\ln(z-c) + \sum_{n=1}^{\infty} \frac{g_n (z-c)^{n+1}}{n+1} \right] .$$

Therefore, when $\lambda_1 = \lambda_2$, the second solution is

$$u = u_1(z) \cdot \phi = A \, u_1(z) + B \, u_1(z) \left[\ln(z-c) + \sum_{n=1}^{\infty} \frac{g_n}{n+1} (z-c)^{n+1} \right] .$$

Thus it is seen that when $\lambda_1 - \lambda_2$ is an integer, the second solution involves a logarithm, except when $g_s = 0$. A practical way of obtaining this second solution is to first obtain the solution $u_1(z)$, and then determine the coefficients in a function

$$\bar{u}_1(z) = \sum_{n=0}^{\infty} b_n (z-c)^{\lambda_2 + n} ,$$

by substituting

$$u = u_1(z) \ln(z-c) + \bar{u}_1(z)$$

in the equation and equating to zero the coefficients of the various powers of z-c in the resulting expression.

Example: Find the solutions of the equation

$$u'' + \frac{1}{z} u' - m^2 u = 0$$

regular near $z = 0$ [8, p. 201].

Equation (1-2) with $c = 0$ becomes

(1-15) $$z^2 u'' + zP(z)\, u' + Q(z)\, u = 0.$$

Thus, if we multiply our given equation by z^2 we have

$$z^2 u'' + zu' - m^2 z^2\, u = 0,$$

where $P(z) = 1$ and $Q(z) = -m^2 z^2$.

Assuming

$$u = z^{\alpha} \sum_{n=0}^{\infty} a_n z^n ,$$

then

$$\begin{array}{l|ll} -m^2z^2 & u = a_o z^{\alpha} + \ldots + a_{2n} z^{\alpha+2n} + a_{2n+2} z^{\alpha+2n+2} + \ldots, & a_o \neq 0. \\ z & u' = a_o \alpha z^{\alpha-1} + \ldots + (\alpha+2n+2) a_{2n+2} z^{\alpha+2n+1} + \ldots. & \\ z^2 & u'' = a_o \alpha(\alpha-1) z^{\alpha-2} + \ldots + (\alpha+2n+2)(\alpha+2n+1) a_{2n+2} z^{\alpha+2n} + \ldots. & \end{array}$$

Equating the coefficients of powers of z-c to zero,

z^{α}: $$\alpha(\alpha-1) + \alpha = 0,$$

$$\alpha^2 = 0 \text{ and } \alpha = 0,0.$$

$z^{\alpha+2n+2}$: $$-m^2 a_{2n} + (\alpha+2n+2) a_{2n+2} + (\alpha+2n+2)(\alpha+2n+1) a_{2n+2} = 0.$$

$$a_{2n+2} = \frac{m^2}{(\alpha+2n+2)^2} ,$$

$$a_{2n+2} = \frac{m^2 a_{2n}}{2^2(n+1)^2} .$$

If n = 0,

$$a_2 = \frac{m^2 a_o}{2^2 1^2} .$$

If n = 1,

$$a_4 = \frac{m^2 a_2}{2^2(2)^2} = \frac{m^4 a_o}{2^4(2!)^2} .$$

If n = 2,

$$a_6 = \frac{m^2 a_4}{2^2(3)^2} = \frac{m^6 a_o}{2^6(3!)^2} .$$

... ...

$$a_{2n} = \frac{m^{2n} a_o}{2^{2n}(n!)^2} , \qquad n=1,2,3,\ldots .$$

Therefore, if we arbitrarily choose $a_o = 1$,

$$u_1 = \sum_{n=o}^{\infty} \frac{m^{2n} z^{2n}}{2^{2n}(n!)^2} .$$

For the second solution, let

$$u = u_1 \ln z + \sum_{n=o}^{\infty} b_n z^n$$

and substitute this into (1-15).

$$u' = u_1' \ln z + u_1 \left(\frac{1}{z} \right) + \sum_{n=1}^{\infty} n b_n z^{n-1} .$$

$$u'' = u_1'' \ln z + 2u_1' \left(\frac{1}{z} \right) - u_1 \left(\frac{1}{z^2} \right) + \sum_{n=2}^{\infty} n(n-1) b_n z^{n-2} .$$

$$u_1'' z^2 \ln z + 2u_1' z - u_1' + \sum_{n=2}^{\infty} n(n-1) b_n z^n + u_1' z \ln z + u_1 + \sum_{n=1}^{\infty} n b_n z^n$$

$$- u_1 m^2 z^2 \ln z - m^2 \sum_{n=0}^{\infty} b_n z^{n+2} = 0.$$

Since the coefficient of ln z is

$$z^2 u_1'' + u_1' z - m^2 z^2 u_1 = 0,$$

we have left

$$2u_1' z + \sum_{n=1}^{\infty} n^2 b_n z^n - m^2 \sum_{n=0}^{\infty} b_n z^{n+2} = 0,$$

$$2u_1' z + \sum_{n=0}^{\infty} (n^2 - m^2 z^2) b_n z^n = 0.$$

Now, taking the derivative of our first solution u_1, we obtain

$$u_1' = \sum_{n=1}^{\infty} \frac{2n \, m^{2n} z^{2n-1}}{2^{2n} (n!)^2} ,$$

so that

$$\sum_{n=1}^{\infty} \frac{4n\, m^{2n} z^{2n}}{2^{2n}(n!)^2} + \sum_{n=o}^{\infty} (n^2 - m^2 z^2) b_n z^n = 0.$$

Equating to zero the coefficients of like powers of z, we have

z^o: $\quad b_o \cdot 0 = 0$, b_o = arbitrary constant.

z^1: $\quad b_1 = 0.$

z^{2n}: $\quad \frac{4n\, m^{2n}}{2^{2n}(n!)^2} + 4n^2 b_{2n} - m^2 b_{2n-2} = 0.$

$$b_{2n} = \frac{1}{n^2}\left[\frac{m^2 b_{2n-2}}{4} - \frac{n\, m^{2n}}{2^{2n}(n!)^2}\right], \quad n=1,2,3,\ldots.$$

Choose $b_o = 0$. Then for $n = 1$,

$$b_2 = \frac{m^2 b_o}{4} - \frac{m^2}{2^2(1!)^2} = \frac{-m^2}{2^2(1!)^2}.$$

If $n = 2$,

$$b_4 = \frac{1}{2^2}\left[\frac{m^2}{4} b_2 - \frac{2m^4}{2^4(2!)^2}\right]$$

$$= -\frac{m^4}{2^2}\left[\frac{1}{2^4(1!)^2} + \frac{1}{2^4 \cdot 2^2}\right],$$

$$b_4 = \frac{-m^4}{2^4(2!)^2}\left(1 + \frac{1}{2}\right).$$

If $n = 3$,

$$b_6 = \frac{1}{3^2}\left[\frac{m^2 b_4}{4} - \frac{3m^6}{2^6(3!)^2}\right]$$

$$= \frac{-1}{3^2}\left[\frac{m^6}{2^6(2!)^2}\left(1+\frac{1}{2}\right)+\frac{3m^6}{2^6(3!)^2}\right],$$

$$b_6 = \frac{-m^6}{2^6(3!)^2}\left(1+\frac{1}{2}+\frac{1}{3}\right).$$

...

Finally,

$$b_{2n} = \frac{-m^{2n}}{2^{2n}(n!)^2}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right) = -\frac{m^{2n}H_n}{2^{2n}(n!)^2},$$

$$H_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}.$$

For n odd,

z^n: $$n^2b_n - m^2b_{n-2} = 0,$$

$$b_n = \frac{m^2}{n^2}b_{n-2} \equiv 0, \qquad n \text{ odd},$$

since $b_1 = 0$.

Therefore, the second solution is

$$u(z) = u_1 \ln z - \sum_{n=1}^{\infty}\frac{m^{2n}H_n z^{2n}}{2^{2n}(n!)^2}.$$

1.5 Solutions Valid for Large Values of $|z|$. Let us now consider solutions valid for large values of $|z|$. To do this we let $z = \frac{1}{z_1}$ in equation (1-1). If the solution of the transformed equation is valid for sufficiently small values of $|z_1|$, then the original equation is said to be valid for

large values of $|z|$. If the point $z_1 = 0$ is an ordinary (or regular or irregular) point of the transformed equation, then the point at infinity is said to be an ordinary (or regular or irregular) point of the original equation.

We begin with the change of variable $z = \frac{1}{z_1}$ in the equation

$$\frac{d^2u}{dz^2} + p(z)\frac{du}{dz} + q(z)\,u = 0. \tag{1-1}$$

$$z = \frac{1}{z_1}; \qquad z_1 = \frac{1}{z}; \qquad dz_1 = -\frac{1}{z^2}\,dz.$$

$$\frac{du}{dz} = \frac{du}{dz_1}\cdot\frac{dz_1}{dz} = -\frac{1}{z^2}\frac{du}{dz_1}\cdot$$

$$\frac{d^2u}{dz^2} = \frac{2}{z^3}\frac{du}{dz_1} - \frac{1}{z^2}\frac{d^2u}{dz_1^2}\cdot\frac{dz_1}{dz},$$

$$\frac{d^2u}{dz^2} = 2z_1^3\frac{du}{dz_1} + z_1^4\frac{d^2u}{dz_1^2}\cdot$$

Substituting in (1-1) we obtain

$$z_1^4\frac{d^2u}{dz_1^2} + \left[2z_1^3 - z_1^2\,p\left(\frac{1}{z_1}\right)\right]\frac{du}{dz_1} + q\left(\frac{1}{z_1}\right)u = 0.$$

Dividing by z_1^4

$$\frac{d^2u}{dz_1^2} + \left[\frac{2}{z_1} - p\left(\frac{1}{z_1}\right)\cdot\frac{1}{z_1^2}\right]\frac{du}{dz_1} + q\left(\frac{1}{z_1}\right)\cdot\frac{1}{z_1^4}\,u = 0.$$

Now $z_1 = 0$ (i.e. $z = \infty$) will be an ordinary point when the coefficients of $\frac{du}{dz_1}$ and u are both analytic. This requires p and q to have the following forms:

$$p\left(\frac{1}{z_1}\right) = 2z_1 + A_2 z_1^2 + A_3 a_1^3 + \cdots,$$

$$q\left(\frac{1}{z_1}\right) = B_4 z_1^4 + B_5 z_1^5 + B_6 z_1^6 + \cdots, \qquad B_4 \neq 0.$$

Expanding about $z = \infty$:

$$p(z) = \frac{2}{z} + \frac{A_2}{z^2} + \frac{A_3}{z^3} + \cdots,$$

$$q(z) = \frac{B_4}{z^4} + \frac{B_5}{z^5} + \frac{B_6}{z^6} + \cdots, \qquad B_4 \neq 0.$$

$z_1 = 0$ will be a regular singular point when

$2 - \frac{1}{z_1} p\left(\frac{1}{z_1}\right)$ and $q\left(\frac{1}{z_1}\right) \cdot \frac{1}{z_1^2}$ are analytic at $z_1 = 0$.

This means that p and q must be:

$$p\left(\frac{1}{z_1}\right) = C_1 z_1 + C_2 z_1^2 + C_3 z_1^3 + \cdots, \qquad C_1 \neq 0.$$

$$q\left(\frac{1}{z_1}\right) = D_2 z_1^2 + D_3 a_1^3 + D_4 z_1^4 + \cdots, \qquad D_2 \neq 0.$$

Expanding about $z = \infty$:

$$p(z) = \frac{C_1}{z} + \frac{C_2}{z^2} + \frac{C_3}{z^3} + \cdots, \qquad C_1 \neq 0.$$

$$q(z) = \frac{D_2}{z^2} + \frac{D_3}{z^3} + \frac{D_4}{z^4} + \cdots, \qquad D_2 \neq 0.$$

Further, it might be helpful to know for what values of ℓ, m and n $z = \infty$ will be a regular point in the following equation having polynomial coefficients ($p_o \neq 0$ and ℓ, m, n non-negative integers).

$$(p_o z^{\ell} + p_1 z^{\ell-1} + \ldots + p_{\ell})\, u'' + (q_o z^m + q_1 z^{m-1} + \ldots + q_m)\, u'$$

$$+ (r_o z^n + r_1 z^{n-1} + \ldots + r_n)\, u = 0.$$

Dividing by the coefficient of u",

$$u'' + \left(\frac{q_o}{p_o} z^{m-\ell} + E_1 z^{m-\ell-1} + \ldots\right) u' + \left(\frac{r_o}{p_o} z^{n-\ell} + F_1 z^{n-\ell-1} + \ldots\right) u = 0.$$

We must have $m-\ell = -1$ or $\ell = m+1$ and $n-\ell = -2$, $\ell = n+2$. Therefore, if $z = \infty$ is a regular singular point, then $\ell = m+1 = n+2$.

1.6 Irregular Singular Points and Confluence. We have already shown in section 1.2 that near an irregular singular point a second order differential equation cannot have two solutions because the indicial equation is at most of the first degree; there may be one or no solutions of this form near this point.

If a differential equation A is obtained from another equation B by making two or more singularities of equation B tend to coincidence, such a limiting process is called confluence. Equation A is called a confluent form of equation B.

CHAPTER II

Confluent Differential Equations

2.1 Differential Equation with Four Singular Points. We shall now develop a general form for a second order linear differential equation which has every point except a_1, a_2, a_3 and a_4 as ordinary points ($z = \infty$ being an ordinary point). These four points are to be regular singular points and let their exponents be α_r, β_r at a_r $(r=1,2,3,4)$. Then the form of our equation will have to be

$$\frac{d^2u}{dz^2} + \left[\sum_{r=1}^{4} \frac{k_r}{z-a_r} + H(z)\right]\frac{du}{dz} + \left\{\sum_{r=1}^{4}\left[\frac{\ell_r}{(z-a_r)^2} + \frac{m_r}{z-a_r}\right] + J(z)\right\} u = 0,$$

where k_r, ℓ_r, m_r $(r=1,2,3,4)$ are constants to be determined, and $H(z)$, $J(z)$ are polynomials in z.

Let

$$u = h_o(z-a_1)^{\lambda} + h_1(z-a_1)^{\lambda+1}+\ldots, \qquad h_o \neq 0,$$

then

$$\sum_{r=1}^{4}\left[\frac{\ell_r}{(z-a_r)^2} + \frac{m_r}{z-a_r}\right] + J(z) \;\Big|\; u = h_o(z-a_1)^{\lambda}+ h_1(z-a_1)^{\lambda+1}+\ldots.$$

$$\sum_{r=1}^{4} \frac{k_r}{z-a_r} + H(z) \;\Big|\; u' = h_o\lambda(z-a_1)^{\lambda-1}+\ldots.$$

$$1 \;\Big|\; u'' = h_o\lambda(\lambda-1)(z-a_1)^{\lambda-2}+\ldots.$$

The indicial equation for $z = a_1$ is

$$(z-a_1)^{\lambda-2}: \qquad h_o\lambda(\lambda-1) + h_o k_1\lambda + h_o \ell_1 = 0,$$

$$\lambda^2 + (k_1-1)\lambda + \ell_1 = 0.$$

Then

$$\lambda = \alpha_1, \beta_1$$

so that

$$\alpha_1 + \beta_1 = 1-k_1, \qquad k_1 = 1 - \alpha_1 - \beta_1,$$

and

$$\alpha_1\beta_1 = \ell_1.$$

Similarly for the other regular points, so that

$$(2\text{-}1) \qquad \frac{d^2u}{dz^2} + \left[\sum_{r=1}^{4} \frac{1-\alpha_r-\beta_r}{z-a_r} + H(z) \right] \frac{du}{dz}$$

$$+ \left\{ \sum_{r=1}^{4} \left[\frac{\alpha_r\beta_r}{(z-a_r)^2} + \frac{m_r}{z-a_r} \right] + J(z) \right\} u = 0.$$

If $z = \infty$ (or $z_1 = 0$, $z = \frac{1}{z_1}$) is to be an ordinary point,

$2z-z^2p(z)$ and $z^4q(z)$ should be analytic at $z = \infty$,

or $\frac{2}{z_1} - \frac{1}{z_1^2}\, p\left(\frac{1}{z_1}\right)$ and $\frac{1}{z_1^4}\, q\left(\frac{1}{z_1}\right)$ should be analytic at $z_1 = 0$

(section 1.5).

Now

$$\frac{2}{z_1} - \frac{1}{z_1^2}\, p\left(\frac{1}{z_1}\right) = \frac{2}{z_1} - \frac{1}{z_1^2} \sum_{r=1}^{4} \frac{1-\alpha_r-\beta_r}{\frac{1}{z_1}-a_r} \frac{1}{z_1^2}\, H\left(\frac{1}{z_1}\right)$$

$$= \frac{2}{z_1} - \sum_{r=1}^{4} \frac{1-\alpha_r-\beta_r}{z_1(1-a_r z_1)} - \frac{1}{z_1^2} H\left(\frac{1}{z_1}\right)$$

$$= \frac{2}{z_1} - \sum_{r=1}^{4} \frac{(1-\alpha_r-\beta_r)}{z_1} \left(\frac{1}{1-a_r z_1}\right) - \frac{1}{z_1^2} H\left(\frac{1}{z_1}\right)$$

$$= \frac{2}{z_1} - \sum_{r=1}^{4} \frac{(1-\alpha_r-\beta_r)}{z_1} \left(1+a_r z_1 + a_r^2 z_1^2 + \ldots\right)$$

$$- \frac{1}{z_1^2} H\left(\frac{1}{z_1}\right)$$

$$= \frac{2 - \sum_{r=1}^{4} (1-\alpha_r-\beta_r)}{z_1} + \sum_{n=o}^{\infty} h_n z_1^n - \frac{1}{z_1^2} H\left(\frac{1}{z_1}\right),$$

where h_n are constants. When $z_1 = 0$, we need

$$H\left(\frac{1}{z_1}\right) = 0$$

and

$$2 - \sum_{r=1}^{4} (1-\alpha_r-\beta_r) = 0,$$

or

$$\sum_{r=1}^{4} (\alpha_r+\beta_r) = 2.$$

Rewriting q(z) in the form

$$q(z) = \frac{n_1}{(z-a_1)^2(z-a_2)(z-a_3)} + \frac{n_2}{(z-a_2)^2(z-a_3)(z-a_4)}$$

$$+ \frac{n_3}{(z-a_3)^2(z-a_4)(z-a_1)} + \frac{n_4}{(z-a_4)^2(z-a_1)(z-a_2)} + J(z),$$

then

$$\frac{1}{z_1^4} q\left(\frac{1}{z_1}\right) = \frac{n_1}{(1-a_1z_1)^2(1-a_2z_1)(1-a_3z_1)}$$

$$+ \frac{n_2}{(1-a_2z_1)^2(1-a_3z_1)(1-a_4z_1)} + \frac{n_3}{(1-a_3z_1)^2(1-a_4z_1)(1-a_1z_1)}$$

$$+ \frac{n_4}{(1-a_4z_1)^2(1-a_1z_1)(1-a_2z_1)} - \frac{1}{z_1^4} J\left(\frac{1}{z_1}\right),$$

which is analytic at $z_1 = 0$, if $J\left(\frac{1}{z_1}\right) = 0$.

To evaluate n_r $(r=1,2,3,4)$, we have

$$\frac{n_1}{(z-a_1)^2(z-a_2)(z-a_3)} + \frac{n_2}{(z-a_2)^2(z-a_3)(z-a_4)}$$

$$+ \frac{n_3}{(z-a_3)^2(z-a_4)(z-a_1)} + \frac{n_4}{(z-a_4)^2(z-a_1)(z-a_2)}$$

$$\equiv \sum_{r=1}^{4} \left[\frac{\alpha_r\beta_r}{(z-a_r)^2} + \frac{m_r}{z-a_r}\right].$$

Then clearing of fractions,

$$n_1(z-a_2)(z-a_3)(z-a_4)^2 + n_2(z-a_1)^2(z-a_3)(z-a_4)$$

$$+ n_3(z-a_1)(z-a_2)^2(z-a_4) + n_4(z-a_1)(z-a_2)(z-a_3)^2$$

$$\equiv \alpha_1\beta_1(z-a_2)^2(z-a_3)^2(z-a_4)^2 + \alpha_2\beta_2(z-a_1)^2(z-a_3)^2(z-a_4)^2$$

$$+\alpha_3\beta_3(z-a_1)^2(z-a_2)^2(z-a_4)^2 + \alpha_4\beta_4(z-a_1)^2(z-a_2)^2(z-a_3)^2$$

$$+ m_1(z-a_1)(z-a_2)^2(z-a_3)^2(z-a_4)^2$$

$$+ m_2(z-a_1)^2(z-a_2)(z-a_3)^2(z-a_4)^2$$

$$+ m_3(z-a_1)^2(z-a_2)^2(z-a_3)(z-a_4)^2$$

$$+ m_4(z-a_1)^2(z-a_2)^2(z-a_3)^2(z-a_4).$$

To determine n_1, let $z = a_1$:

$$n_1(a_1-a_2)(a_1-a_3)(a_1-a_4)^2 \equiv \alpha_1\beta_1(a_1-a_2)^2(a_1-a_3)^2(a_1-a_4)^2,$$

$$n_1 \equiv \alpha_1\beta_1(a_1-a_2)(a_1-a_3).$$

Similarly,

$$n_2 = \alpha_2\beta_2(a_2-a_3)(a_2-a_4),$$

$$n_3 = \alpha_3\beta_3(a_3-a_4)(a_3-a_1),$$

$$n_4 = \alpha_4\beta_4(a_4-a_1)(a_4-a_2).$$

Therefore, the general form for a second order linear differential equation with regular singular points at $z = a_r$ $(r= 1,2,3,4)$ is

$$(2\text{-}2) \qquad \frac{d^2u}{dz^2} + \sum_{r=1}^{4} \frac{1-\alpha_r-\beta_r}{z-a_r} \frac{du}{dz} + \left[\frac{\alpha_1\beta_1(a_1-a_2)(a_1-a_3)}{(z-a_1)^2(z-a_2)(z-a_3)} \right.$$

$$+ \frac{\alpha_2\beta_2(a_2-a_3)(a_2-a_4)}{(z-a_2)^2(z-a_3)(z-a_4)} + \frac{\alpha_3\beta_3(a_3-a_4)(a_3-a_1)}{(z-a_3)^2(z-a_4)(z-a_1)}$$

$$\left. + \frac{\alpha_4\beta_4(a_4-a_1)(a_4-a_2)}{(z-a_4)^2(z-a_1)(z-a_2)} \right] u = 0,$$

where

$$\sum_{r=1}^{4} (\alpha_r+\beta_r) = 2,$$

α_r and β_r being the exponents at $z = a_r$.

To express the fact that u satisfies an equation of this type we will write

$$(2\text{-}3) \qquad u = \left\{ \begin{matrix} a_1 & a_2 & a_3 & a_4 & \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & z \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \end{matrix} \right\} .$$

2.2 Differential Equation with Singular Points at $z = 0, 1, \infty, \infty$. Let us now consider the confluent case where $a_2 = 0$, $a_3 = 1$, $a_1 = a_4$ become infinite.

First, let $a_2 = 0$ and $a_3 = 1$, then (2-2) becomes

$$(2\text{-}4)\quad \frac{d^2u}{dz^2} + \left[\frac{1-\alpha_1-\beta_1}{z-a_1} + \frac{1-\alpha_2-\beta_2}{z} + \frac{1-\alpha_3-\beta_3}{z-1} + \frac{1-\alpha_4-\beta_4}{z-a_4}\right]\frac{du}{dz}$$

$$+ \left[\frac{\alpha_1\beta_1 a_1(a_1-1)}{(z-a_1)^2 z(z-1)} + \frac{\alpha_2\beta_2 a_4}{z^2(z-1)(z-a_4)} + \frac{\alpha_3\beta_3(1-a_4)(1-a_1)}{(z-1)^2(z-a_4)(z-a_1)}\right.$$

$$\left. \frac{\alpha_4\beta_4(a_4-a_1)a_4}{(z-a_4)^2(z-a_1)z}\right] u = 0.$$

Before letting a_4 become infinite, rewrite the equation in the form

$$\frac{d^2u}{dz^2} + \left[\frac{1-\alpha_1-\beta_1}{z-a_1} + \frac{1-\alpha_2-\beta_2}{z} + \frac{1-\alpha_3-\beta_3}{z-1} + \frac{1-\alpha_4-\beta_4}{z-a_4}\right]\frac{du}{dz}$$

$$+ \left[\frac{\alpha_1\beta_1 a_1(a_1-1)}{(z-a_1)^2 z(z-1)} + \frac{\alpha_2\beta_2}{z^2(z-1)\left(\frac{z}{a_4}-1\right)} + \frac{\alpha_3\beta_3\left(\frac{1}{a_4}-1\right)(1-a_1)}{(z-1)^2\left(\frac{z}{a_4}-1\right)(z-a_1)}\right.$$

$$\left. + \frac{\alpha_4\beta_4\left(1-\frac{a_1}{a_4}\right)}{\left(\frac{z}{a_4}-1\right)^2(z-a_1)z}\right] u = 0.$$

Now let a_4 become infinite:

$$(2\text{-}5)\quad \frac{d^2u}{dz^2} + \left[\frac{1-\alpha_1-\beta_1}{z-a_1} + \frac{1-\alpha_2-\beta_2}{z} + \frac{1-\alpha_3-\beta_3}{z-1}\right]\frac{du}{dz}$$

$$+\left[\frac{\alpha_1\beta_1 a_1(a_1-1)}{(z-a_1)^2 z(z-1)} - \frac{\alpha_2\beta_2}{z^2(z-1)} + \frac{\alpha_3\beta_3(1-a_1)}{(z-1)^2(z-a_1)} + \frac{\alpha_4\beta_4}{(z-a_1)z}\right]u = 0,$$

which may be represented by

$$(2\text{-}6) \qquad u = \left\{ \begin{matrix} a_1 & 0 & 1 & \infty & \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & z \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \end{matrix} \right\} .$$

This equation is actually more general then one might assume, for by means of the linear transformation

$$u = \frac{(a_4-a_3)}{(a_2-a_3)} \cdot \frac{(z-a_2)}{(z-a_4)} ,$$

any three points $z = a_2, a_3, a_4$ can be carried into $u = 0, 1, \infty$.

Finally, we want to let a_1 become infinite to obtain the confluent case with singular points $0, 1, \infty, \infty$. Rewrite the equation (2-5) in the form

$$\frac{d^2u}{dz^2} + \left[\frac{1-\alpha_1-\beta_1}{z-a_1} + \frac{1-\alpha_2-\beta_2}{z} + \frac{1-\alpha_3-\beta_3}{z-1} \right] \frac{du}{dz}$$

$$+ \left[\frac{\alpha_1\beta_1\left(1-\frac{1}{a_1}\right)}{\left(\frac{z}{a_1}-1\right)^2 z(z-1)} - \frac{\alpha_2\beta_2}{z^2(z-1)} + \frac{\alpha_3\beta_3\left(\frac{1}{a_1}-1\right)}{(z-1)^2\left(\frac{z}{a_1}-1\right)} \right.$$

$$\left. + \frac{\alpha_4\beta_4}{(z-a_1)z} \right] u = 0,$$

and when a_1 becomes infinite we have the confluent equation

$$(2\text{-}7)\quad \frac{d^2u}{dz^2} + \left[\frac{1-\alpha_2-\beta_2}{z} + \frac{1-\alpha_3-\beta_3}{z-1}\right]\frac{du}{dz} + \left[\frac{\alpha_1\beta_1}{z(z-1)} - \frac{\alpha_2\beta_2}{z^2(z-1)} + \frac{\alpha_3\beta_3}{(z-1)^2}\right] u = 0,$$

with singular points at $z = 0,\ 1,\ \infty,\ \infty$.

Let us proceed to find solutions of this equation. Multiplying the equation by $z^2(z-1)^2$, we have

$$(2\text{-}8)\quad z^2(z-1)^2\frac{d^2u}{dz^2} + \left[(2-\alpha_2-\beta_2-\alpha_3-\beta_3)z^3 + (-3+2\alpha_2+2\beta_2+\alpha_3+\beta_3)z^2 + (1-\alpha_2-\beta_2)z\right]\frac{du}{dz} + \left[(\alpha_1\beta_1 + \alpha_3\beta_3)z^2 + (-\alpha_1\beta_1 - \alpha_2\beta_2)z + \alpha_2\beta_2\right] u = 0.$$

Assume the solution

$$u = C_o z^{\lambda} + C_1 z^{\lambda+1} + \ldots + C_{n-2} z^{\lambda+n-2} + C_{n-1} z^{\lambda+n-1} + C_n z^{\lambda+n} + \ldots, \qquad C_o \neq 0,$$

then

$$u' = C_o\lambda z^{\lambda-1} + C_1(\lambda+1)z^{\lambda} + \ldots + C_{n-2}(\lambda+n-2)z^{\lambda+n-3} + C_{n-1}(\lambda+n-1)z^{\lambda+n-2} + C_n(\lambda+n)z^{\lambda+n-1} + \ldots,$$

$$u'' = C_o\lambda(\lambda-1)z^{\lambda-2} + \ldots + C_{n-2}(\lambda+n-2)(\lambda+n-3)z^{\lambda+n-4}$$

$$+ C_{n-1}(\lambda+n-1)(\lambda+n-2)z^{\lambda+n-3} + C_n(\lambda+n)(\lambda+n-1)z^{\lambda+n-2}+\ldots.$$

If these equations for u and its derivatives were now substituted in equation (2-8) and the coefficients of the powers of z equated to zero, we would obtain the following results.

The indicial equation is

z^{λ}:
$$\lambda(\lambda-1) + \lambda(1-\alpha_2-\beta_2) + \alpha_2\beta_2 = 0.$$

(2-9)
$$\lambda^2 - (\alpha_2+\beta_2)\lambda + \alpha_2\beta_2 = 0.$$

Then

$$\lambda = \alpha_2, \beta_2.$$

Finding C_n,

$z^{\lambda+n}$:
$$C_{n-2}(\lambda+n-2)(\lambda+n-3) - 2\,C_{n-1}(\lambda+n-1)(\lambda+n-2)$$

$$+ C_n(\lambda+n)(\lambda+n-1) + C_{n-2}(\lambda+n-2)(2-\alpha_2-\beta_2-\alpha_3-\beta_3)$$

$$+ C_{n-1}(\lambda+n-1)(-3+2\alpha_2+2\beta_2+\alpha_3+\beta_3) + C_n(\lambda+n)(1-\alpha_2-\beta_2)$$

$$+ C_{n-2}(\alpha_1\beta_1+\alpha_3\beta_3) + C_{n-1}(-\alpha_1\beta_1-\alpha_2\beta_2) + C_n\alpha_2\beta_2 = 0.$$

$$C_n = \Big\{ C_{n-2}\Big[-(\lambda+n-2)(\lambda+n-3)-(\lambda+n-2)(2-\alpha_2-\beta_2-\alpha_3-\beta_3)-(\alpha_1\beta_1+\alpha_3\beta_3)\Big]$$

$$+ C_{n-1}\Big[2(\lambda+n-1)(\lambda+n-2)-(\lambda+n-1)(-3+2\alpha_2+2\beta_2+\alpha_3+\beta_3)$$

$$+ \alpha_1\beta_1 + \alpha_2\beta_2\Big]\Big\} \frac{1}{(\lambda+n)(\lambda+n-1)+(\lambda+n)(1-\alpha_2-\beta_2) + \alpha_2\beta_2}.$$

Simplifying the denominator by using the indicial equation (2-9) we have

$$C_n = \left\{ C_{n-2}\left[-(\lambda+n-2)(\lambda+n-1-\alpha_2-\beta_2-\alpha_3-\beta_3) - \alpha_1\beta_1 - \alpha_3\beta_3 \right] + C_{n-1}\left[(\lambda+n-1)(2\lambda+2n-1-2\alpha_2-2\beta_2-\alpha_3-\beta_3) + \alpha_1\beta_1 + \alpha_2\beta_2 \right] \right\} \frac{1}{n(2\lambda+n-\alpha_2-\beta_2)} .$$

Let us compute a few terms of our series solution to see if a general term can be obtained.

$$C_1 = \frac{C_o\left[\lambda(2\lambda+1-2\alpha_2-2\beta_2-\alpha_3-\beta_3) + \alpha_1\beta_1 + \alpha_2\beta_2 \right]}{(2\lambda+1-\alpha_2-\beta_2)} .$$

Using the indicial equation (2-9) we can simplify and write

$$C_1 = \frac{C_o\left[(1-\alpha_3-\beta_3)\lambda + \alpha_1\beta_1 - \alpha_2\beta_2 \right]}{2\lambda+1-\alpha_2-\beta_2} .$$

$$C_2 = \left\{ C_o\left[-\lambda(\lambda+1-\alpha_2-\beta_2-\alpha_3-\beta_3) - \alpha_1\beta_1 - \alpha_3\beta_3 \right] + C_1\left[(\lambda+1)(2\lambda+3-2\alpha_2-2\beta_2-\alpha_3-\beta_3) + \alpha_1\beta_1 + \alpha_2\beta_2 \right] \right\} \frac{1}{2(2\lambda+2-\alpha_2-\beta_2)} .$$

Substituting C_1 into this expression:

$$C_2 = C_o \left\{ (2\lambda+1-\alpha_2-\beta_2)\left[-\lambda(\lambda+1-\alpha_2-\beta_2-\alpha_3-\beta_3)-\alpha_1\beta_1-\alpha_3\beta_3 \right] \right.$$

$$+\left[(1-\alpha_3-\beta_3)\lambda + \alpha_1\beta_1-\alpha_2\beta_2 \right] \left[(\lambda+1)(2\lambda+3-2\alpha_2-2\beta_2-\alpha_3-\beta_3) \right.$$

$$\left. \left. + \alpha_1\beta_1 + \alpha_2\beta_2 \right] \right\} \cdot \frac{1}{2(2\lambda+1-\alpha_2-\beta_2)(2\lambda+2-\alpha_2-\beta_2)} \quad .$$

It seems apparent from the appearance of C_1 and C_2 and from the fact that C_n is a three term recurrence relation, that a greal deal of work might be required to write these two solutions of (2-7) with a general term, thereby making available a form suitable for further development. Since equation (2-7) has five undetermined parameters ($\alpha_1\beta_1, \alpha_2, \beta_2$ α_3, β_3), let us consider the conditions necessary to insure a solution with a two term recurrence relation. This can be readily accomplished by applying Scheffé's Criteria [6, p. 240] which Crowson has presented and proved in his dissertation [1, p. 114].

2.3 Scheffé's Criteria Applied to Confluent Case. The criteria to be applied is: Necessary and sufficient conditions for a solution of a second order linear ordinary differential equation,

$$p_2(z)u''(z) + p_1(z)u'(z) + p_o(z)u(z) = 0,$$

to have a two term recurrence equation, relative to the point $z = 0$, is that in some neighborhood of the point $z = 0$,

$$p_j(z) = \left[S_j - T_j z^h \right] z^{j-m} , \tag{2-10}$$

where m is an integer, h is a positive integer, and S_j, T_j are constants such that $S_i \, T_j \neq 0$ for some $i = 0,1,2$ and $j = 0,1,2$.

In assigning values to the exponents, it can be seen from equation (2-4) that none of the exponents can be a function of a_4. Since the sum of the exponents must be 2, if there are any exponents which are functions of a_4 there must be at least two such exponents. α_r, β_r $(r = 1,2,3)$ could not be functions of a_4, otherwise the coefficient of $\frac{du}{dz}$ will be undefined when a_4 becomes infinite. If α_4 and β_4 are functions of a_4, then the term

$$\frac{\alpha_4 \beta_4 (a_4 - a_1) a_4}{(z-a_4)^2 (z-a_1) z}$$

would be undefined as a_4 becomes infinite. Thus the exponents can not be functions of a_4.

Next, suppose that two or more exponents in equation (2-6) are functions of a_1. Such an equation might be

$$u = \left\{ \begin{matrix} a_1 & 0 & 1 & \infty & \\ a_1 \ell + f & \alpha_2 & \alpha_3 & -a_1 \ell - \frac{1}{a_1} + t & z \\ \frac{1}{a_1} & \beta_2 & \beta_3 & \beta_4 & \end{matrix} \right\} ,$$

where the sum of the exponents is

$$f + \alpha_2 + \beta_2 + \alpha_3 + \beta_3 + t + \beta_4 = 2,$$

f, ℓ and t are arbitrary constants. Using (2-2) we can write this equation in the form

$$u'' + \left[\frac{1-a_1\ell-f-\frac{1}{a_1}}{z-a_1} + \frac{1-\alpha_2-\beta_2}{z} + \frac{1-\alpha_3-\beta_3}{z-1}\right] u' + \left[\frac{(\ell+\frac{f}{a_1})a_1(a_1-1)}{(z-a_1)^2 z(z-1)}\right.$$

$$\left. - \frac{\alpha_2\beta_2}{z^2(z-1)} + \frac{\alpha_3\beta_3(1-a_1)}{(z-1)^2(z-a_1)} + \frac{\beta_4(-a_1\ell-\frac{1}{a_1}+t)}{(z-a_1)z}\right] u = 0.$$

$$u'' + \left[\frac{\left(\frac{1}{a_1} - \ell - \frac{f}{a_1} - \frac{1}{a_1^2}\right)}{\left(\frac{z}{a_1}-1\right)} + \frac{1-\alpha_2-\beta_2}{z} + \frac{1-\alpha_3-\beta_3}{z-1}\right] u'$$

$$+\left[\frac{\left(\ell+\frac{f}{a_1}\right)\left(1-\frac{1}{a_1}\right)}{\left(\frac{z}{a_1}-1\right)^2 z(z-1)} - \frac{\alpha_2\beta_2}{z^2(z-1)} + \frac{\alpha_3\beta_3\left(\frac{1}{a_1}-1\right)}{(z-1)^2\left(\frac{z}{a_1}-1\right)}\right.$$

$$\left. + \frac{\beta_4\left(-\ell-\frac{1}{a_1^2}+\frac{t}{a_1}\right)}{\left(\frac{z}{a_1}-1\right)z}\right] u = 0.$$

Letting a_1 become infinite:

$$u'' + \left[\ell + \frac{1-\alpha_2-\beta_2}{z} + \frac{1-\alpha_3-\beta_3}{z-1}\right] u'$$

$$+ \left[\frac{\ell}{z(z-1)} - \frac{\alpha_2\beta_2}{z^2(z-1)} + \frac{\alpha_3\beta_3}{(z-1)^2} + \frac{\beta_4\ell}{z}\right] u = 0.$$

Clearing of fractions we can write

$$(2\text{-}11)\qquad z^2(z-1)^2u'' + \left[\ell z^2(z-1)^2+(1-\alpha_2-\beta_2)z(z-1)^2 + (1-\alpha_3-\beta_3)z^2(z-1)\right] u'$$

$$+\left[\ell z(z-1)-\alpha_2\beta_2(z-1) + \alpha_3\beta_3 z^2 + \beta_4\ell z(z-1)^2\right] u = 0,$$

where the singular points are at $z = 0, 1, \infty, \infty$.

Applying the criteria (2-10) we have for $j = 2$

$$p_2 = \left[S_2 - T_2 z^h\right]z^{2-m} = z^2(z-1)^2,$$

$$S_2 z^{2-m} - T_2 z^{h-m+2} = z^4-2z^3 + z^2,$$

but this is impossible. However, this difficulty can be avoided by reducing the coefficient of u'' by a factor of at least $z-1$. To do this $\alpha_3\beta_3$ must equal zero, so let $\beta_3=0$. Thus, with $\beta_3=0$ and dividing equation (2-11) by $z-1$, we now have

$$z^2(z-1)u'' + \left[\ell z^2(z-1)+(1-\alpha_2-\beta_2)z(z-1)+(1-\alpha_3)z^2\right]u'$$

$$+ \left[\ell z-\alpha_2\beta_2 + \beta_4\ell z(z-1)\right] u = 0.$$

$$(z^3-z^2)u'' + \left[\ell z^3 +(2-\alpha_2-\beta_2-\alpha_3-\ell)z^2 + (-1+\alpha_2+\beta_2)z\right] u'$$

$$+ \left[\beta_4\ell z^2 + (1-\beta_4)\ell z - \alpha_2\beta_2\right] u = 0.$$

Now returning to our criteria (2-10),

$$p_2 = S_2 z^{2-m} - T_2 z^{h-m+2} = z^3 - z^2,$$

where $h-m+2 = 3$ and $2-m = 2$. Therefore, $m = 0$, $h = 1$, $S_2 = T_2 = -1$, $S_2 T_2 \neq 0$. We need

$$p_1 = \left[S_1 - T_1 z \right] z = \ell z^3 + (2-\alpha_2-\beta_2-\alpha_3-\ell) z^2 + (-1+\alpha_2+\beta_2) z,$$

so that $\ell = 0$, $T_1 = \alpha_2+\beta_2 + \alpha_3 - 2$, $S_1 = \alpha_2+\beta_2-1$ and

$$p_1 = (2-\alpha_2-\beta_2-\alpha_3) z^2 + (\alpha_2+\beta_2-1) z.$$

Finally, we have

$$p_o = \left[S_o - T_o z \right] = -\alpha_2\beta_2,$$

where $S_o = -\alpha_2\beta_2$ and $T_o = 0$.

Thus, if $\beta_3 = 0$ in equation (2-11) the resulting equation, which has a two term recurrence relation for a solution, is

$$(2\text{-}12) \quad z^2(z-1)u'' + \left[(2-\alpha_2-\beta_2-\alpha_3) z^2 + (\alpha_2+\beta_2-1) z \right] u' - \alpha_2\beta_2 \, u = 0.$$

The requirement that $f + \alpha_2+\beta_2+\alpha_3+t+\beta_4 = 2$ is actually not restrictive, since f, t and β_4 do not appear in our final result.

Consider the alternative of having the exponents in (2-6) be functions of a_1 and assume that neither a_1 nor a_4 appear in the exponents. Apply the criteria (2-10) to equation

$$(2\text{-}7)\quad u'' + \left[\frac{1-\alpha_2-\beta_2}{z} + \frac{1-\alpha_3-\beta_3}{z-1}\right] u' + \left[\frac{\alpha_1\beta_1}{z(z-1)} - \frac{\alpha_2\beta_2}{z^2(z-1)} + \frac{\alpha_3\beta_3}{(z-1)^2}\right] u = 0,$$

or clearing of fractions

$$z^2(z-1)^2\, u'' + \left[(1-\alpha_2-\beta_2)z(z-1)^2+(1-\alpha_3-\beta_3)z^2(z-1)\right] u' + \left[\alpha_1\beta_1 z(z-1) - \alpha_2\beta_2(z-1) + \alpha_3\beta_3 z^2\right] u = 0,$$

where

$$\sum_{r=1}^{4} (\alpha_r+\beta_r) = 2.$$

As before we must remove a factor of z-1, therefore let $\beta_3 = 0$ so that $\alpha_3\beta_3 = 0$. Then

$$z^2(z-1)u'' + \left[(1-\alpha_2-\beta_2)z(z-1)+(1-\alpha_3)z^2\right]u' + \left[\alpha_1\beta_1 z-\alpha_2\beta_2\right] u = 0,$$

$$(z^3-z^2)u'' + \left[(2-\alpha_2-\beta_2-\alpha_3)z^2+(\alpha_2+\beta_2-1)z\right]u' + \left[\alpha_1\beta_1 z-\alpha_2\beta_2\right]u = 0.$$

Applying the criteria (2-10) we have when $j = 2$

$$p_2 = \left[S_2-T_2 z^h\right]z^{2-m} = z^3-z^2,$$

where $h-m+2 = 3$ and $2-m = 2$. Thence, $m = 0$, $h = 1$, $S_2 = T_2 = -1$, $S_2T_2 \neq 0$. Further,

$$p_1 = \left[S_1-T_1 z\right]z = (2-\alpha_2-\beta_2-\alpha_3)z^2+(\alpha_2+\beta_2-1)z$$

and $T_1 = \alpha_2+\beta_2 + \alpha_3 - 2$, $S_1 = \alpha_2+\beta_2 - 1$. Finally,

$$p_o = S_o - T_o z = \alpha_1\beta_1 z - \alpha_2\beta_2$$

with $T_o = -\alpha_1\beta_1$, $S_o = -\alpha_2\beta_2$.

The equation with a solution having a two term recurrence relation is, therefore,

$$(2\text{-}13)\quad z^2(z-1)u'' + \left[(2-\alpha_2-\beta_2-\alpha_3)z^2 + (\alpha_2+\beta_2-1)z\right] u' + \left[\alpha_1\beta_1 z - \alpha_2\beta_2\right] u = 0,$$

where the only restrictions are that in equation (2-7) α_r, β_r $(r=1,2,3,4)$ contain no function of a_1 or a_4, and $\beta_3 = 0$.

Since equation (2-12) can be obtained from (2-13) by letting $\alpha_1\beta_1 = 0$, let us obtain solutions of the latter.

$$\begin{array}{l|l}
\alpha_1\beta_1 z-\alpha_2\beta_2 & u = C_o z^{\lambda}+\ldots+C_{n-1}z^{\lambda+n-1}+C_n z^{\lambda+n}+\ldots,\ C_o \neq 0, \\
(2-\alpha_2-\beta_2-\alpha_3)z^2 & u' = C_o\lambda z^{\lambda-1}+\ldots+C_{n-1}(\lambda+n-1)z^{\lambda+n-2} \\
+\ (\alpha_2+\beta_2-1)z & \qquad + C_n(\lambda+n)z^{\lambda+n-1}+\ldots, \\
z^3-z^2 & u'' = C_o\lambda(\lambda-1)z^{\lambda-2}+\ldots \\
 & \qquad + C_{n-1}(\lambda+n-1)(\lambda+n-2)z^{\lambda+n-3} \\
 & \qquad + C_n(\lambda+n)(\lambda+n-1)z^{\lambda+n-2}+\ldots.
\end{array}$$

The indicial equation at $z = 0$ is

$$z^{\lambda}: \qquad -\lambda(\lambda-1) + (\alpha_2+\beta_2-1)\lambda - \alpha_2\beta_2 = 0,$$

$$\lambda^2 - (\alpha_2+\beta_2)\lambda + \alpha_2\beta_2 = 0.$$

The exponents are

$$\lambda = \alpha_2, \beta_2 .$$

$z^{\lambda+n}$: $C_{n-1}(\lambda+n-1)(\lambda+n-2)-C_n(\lambda+n)(\lambda+n-1)+C_{n-1}(\lambda+n-1)(2-\alpha_2$

$$-\beta_2-\alpha_3) + C_n(\lambda+n)(\alpha_2+\beta_2-1) + C_{n-1}\alpha_1\beta_1 - C_n\alpha_2\beta_2 = 0.$$

$$C_n = \frac{-C_{n-1}\left[(\lambda+n-1)(\lambda+n-2)+(\lambda+n-1)(2-\alpha_2-\beta_2-\alpha_3) + \alpha_1\beta_1\right]}{-(\lambda+n)(\lambda+n-1)+(\lambda+n)(\alpha_2+\beta_2-1)-\alpha_2\beta_2},$$

$$C_n = \frac{C_{n-1}\left[\alpha_1\beta_1 + (\lambda+n-1)(\lambda+n-\alpha_2-\beta_2-\alpha_3)\right]}{(\lambda+n)^2 - (\alpha_2+\beta_2)(\lambda+n) + \alpha_2\beta_2},$$

$$C_n = \frac{C_{n-1}\left[\alpha_1\beta_1 + (\lambda+n-1)(\lambda+n-\alpha_2-\beta_2-\alpha_3)\right]}{(\lambda+n-\alpha_2)(\lambda+n-\beta_2)}.$$

Let $\lambda = \alpha_2$:

$$C_n = \frac{C_{n-1}\left[\alpha_1\beta_1 + (\alpha_2+n-1)(n-\beta_2-\alpha_3)\right]}{n(\alpha_2-\beta_2+n)}.$$

$$C_1 = \frac{C_o\left[\alpha_1\beta_1 + \alpha_2(1-\beta_2-\alpha_3)\right]}{(\alpha_2-\beta_2+1)}.$$

$$C_2 = \frac{C_1\left[\alpha_1\beta_1 + (\alpha_2+1)(2-\beta_2-\alpha_3)\right]}{2(\alpha_2-\beta_2+2)},$$

$$C_2 = \frac{C_o\left[\alpha_1\beta_1 + \alpha_2(1-\beta_2-\alpha_3)\right]\left[\alpha_1\beta_1 + (\alpha_2+1)(2-\beta_2-\alpha_3)\right]}{2!(\alpha_2-\beta_2+1)(\alpha_2-\beta_2+2)} .$$

Therefore the solutions of (2-13) are:

$$u_1 = C_o z^{\alpha_2}\left[1 + \frac{\alpha_1\beta_1+\alpha_2(1-\beta_2-\alpha_3)}{1!(\alpha_2-\beta_2+1)} z\right.$$

$$+ \frac{[\alpha_1\beta_1+\alpha_2(1-\beta_2-\alpha_3)][\alpha_1\beta_1+(\alpha_2+1)(2-\beta_2-\alpha_3)]}{2!(\alpha_2-\beta_2+1)(\alpha_2-\beta_2+2)} z^2 + \ldots$$

$$\left. + \frac{[\alpha_1\beta_1+\alpha_2(1-\beta_2-\alpha_3)]\ldots[\alpha_1\beta_1+(\alpha_2+n-1)(n-\beta_2-\alpha_3)]}{n!(\alpha_2-\beta_2+1)\ldots(\alpha_2-\beta_2+n)} z^n + \ldots\right].$$

$$(2\text{-}14)\quad u_1 = C_o z^{\alpha_2}\left\{1 + \sum_{n=1}^{\infty} \frac{[\alpha_1\beta_1+\alpha_2(1-\beta_2-\alpha_3)]\ldots[\alpha_1\beta_1+(\alpha_2+n-1)(n-\beta_2-\alpha_3)]}{n!(\alpha_2-\beta_2+1)\ldots(\alpha_2-\beta_2+n)} z^n\right\}$$

and similarly

$$(2\text{-}15)\quad u_2 = C_o z^{\beta_2}\left\{1 + \sum_{n=1}^{\infty} \frac{[\alpha_1\beta_1+\beta_2(1-\alpha_2-\alpha_3)]\ldots[\alpha_1\beta_1+(\beta_2+n-1)(n-\alpha_2-\alpha_3)]}{n!(\beta_2-\alpha_2+1)\ldots(\beta_2-\alpha_2+n)} z^n\right\} .$$

If $\alpha_1\beta_1 = 0$, then the solutions of (2-13) will be hypergeometric:

$$(2\text{-}16)\qquad u_1 = C_o z^{\alpha_2}\ {}_2F_1(\alpha_2, -\beta_2-\alpha_3;\ \alpha_2-\beta_2;\ z)$$

and

$$(2\text{-}17) \qquad u_2 = C_o z^{\beta_2} \; {}_2F_1(\beta_2, -\alpha_2-\alpha_3; \; \beta_2-\alpha_2; \; z) \; .$$

The hypergeometric form is

$$(2\text{-}18) \quad {}_2F_1(a,b; \; c; \; z) = 1 + \frac{a\cdot b}{1!c} z + \frac{a(a+1)b(b+1)}{2!c(c+1)} z^2 + \ldots$$

$$+ \frac{a(a+1)\ldots(a+n-1)b(b+1)\ldots(b+n-1)}{n! \; c(c+1)\ldots(c+n-1)} z^n + \ldots$$

$$= \sum_{n=o}^{\infty} \frac{a(a+1)\ldots(a+n-1)b(b+1)\ldots(b+n-1)}{n! \; c(c+1)\ldots(c+n-1)} z^n \; .$$

2.4 Solutions of Confluent Cases after Normalizing. Another approach to obtaining solutions of equation (2-7), which we will now use, is to remove the u' term and find solutions of this transformed equation.

To normalize the general equation

$$u'' + p(z) \; u' + q(z) \; u = 0,$$

let $u = vw$. Then we have

q	$u = vw.$
p	$u' = v'w + vw'.$
1	$u'' = v''w + 2v'w' + vw''.$

Hence,

$$vw'' + (2v' + vp) \; w' + (v'' + v'p + vq) \; w = 0 \; .$$

Let

$$2v' + vp = 0$$

and solve for v.

$$2\frac{dv}{dz} = -vp,$$

$$\frac{dv}{v} = -\frac{1}{2}p\,dz.$$

Integrating,

$$\ln v = -\frac{1}{2}\int p\,dz.$$

$$v = e^{-\frac{1}{2}\int p\,dz}.$$

Therefore, to normalize the equation

$$u'' + p(z)\,u' + q(z)\,u = 0,$$

let

$$v = e^{-\frac{1}{2}\int p\,dz} \tag{2-19}$$

in

$$vw'' + (v'' + v'p + vq)\,w = 0. \tag{2-20}$$

Now to normalize equation (2-7) we use (2-19) to find

$$v = z^{\frac{1}{2}(\alpha_2+\beta_2-1)}(z-1)^{\frac{1}{2}(\alpha_3+\beta_3-1)}.$$

Then

$$v' = \tfrac{1}{2}(\alpha_2+\beta_2-1)z^{\frac{1}{2}(\alpha_2+\beta_2-3)}(z-1)^{\frac{1}{2}(\alpha_3+\beta_3-1)}$$

$$+ \tfrac{1}{2}(\alpha_3+\beta_3-1)z^{\frac{1}{2}(\alpha_2+\beta_2-1)}(z-1)^{\frac{1}{2}(\alpha_3+\beta_3-3)},$$

$$v' = \frac{1}{2}\, z^{\frac{1}{2}(\alpha_2+\beta_2-3)}\,(z-1)^{\frac{1}{2}(\alpha_3+\beta_3-3)}\left[(\alpha_2+\beta_2-1)(z-1)\right.$$

$$\left. + (\alpha_3+\beta_3-1)z\right] .$$

$$v'' = \tfrac{1}{4}(\alpha_2+\beta_2-1)(\alpha_2+\beta_2-3)z^{\frac{1}{2}(\alpha_2+\beta_2-5)}\,(z-1)^{\frac{1}{2}(\alpha_3+\beta_3-1)}$$

$$+ \tfrac{1}{2}(\alpha_2+\beta_2-1)(\alpha_3+\beta_3-1)z^{\frac{1}{2}(\alpha_2+\beta_2-3)}\,(z-1)^{\frac{1}{2}(\alpha_3+\beta_3-3)}$$

$$+ \tfrac{1}{4}(\alpha_3+\beta_3-1)(\alpha_3+\beta_3-3)z^{\frac{1}{2}(\alpha_2+\beta_2-1)}\,(z-1)^{\frac{1}{2}(\alpha_3+\beta_3-5)} ,$$

$$v'' = \tfrac{1}{2}z^{\frac{1}{2}(\alpha_2+\beta_2-5)}(z-1)^{\frac{1}{2}(\alpha_3+\beta_3-5)}\left[\tfrac{1}{2}(\alpha_2+\beta_2-1)(\alpha_2+\beta_2-3)(z-1)^2\right.$$

$$\left. + (\alpha_2+\beta_2-1)(\alpha_3+\beta_3-1)z(z-1) + \tfrac{1}{2}(\alpha_3+\beta_3-1)(\alpha_3+\beta_3-3)z^2\right] .$$

Substituting v into equation (2-20) with p and q from (2-7),

$$z^{\frac{1}{2}(\alpha_2+\beta_2-1)}\,(z-1)^{\frac{1}{2}(\alpha_3+\beta_3-1)}\,w''$$

$$+\left\{\tfrac{1}{2}z^{\frac{1}{2}(\alpha_2+\beta_2-5)}(z-1)^{\frac{1}{2}(\alpha_3+\beta_3-5)}\left[\tfrac{1}{2}(\alpha_2+\beta_2-1)(\alpha_2+\beta_2-3)(z-1)^2\right.\right.$$

$$\left.\left. + (\alpha_2+\beta_2-1)(\alpha_3+\beta_3-1)z(z-1) + \tfrac{1}{2}(\alpha_3+\beta_3-1)(\alpha_3+\beta_3-3)z^2\right]\right.$$

$$+ \tfrac{1}{2}z^{\frac{1}{2}(\alpha_2+\beta_2-3)}\,(z-1)^{\frac{1}{2}(\alpha_3+\beta_3-3)}\left[(\alpha_2+\beta_2-1)(z-1)\right.$$

$$+ (\alpha_3+\beta_3-1)z\Big]\left(\frac{1-\alpha_2-\beta_2}{z} + \frac{1-\alpha_3-\beta_3}{z-1}\right)$$

$$+ z^{\frac{1}{2}(\alpha_2+\beta_2-1)}(z-1)^{\frac{1}{2}(\alpha_3+\beta_3-1)}\left[\frac{\alpha_1\beta_1}{z(z-1)} - \frac{\alpha_2\beta_2}{z^2(z-1)}\right.$$

$$\left.+ \frac{\alpha_3\beta_3}{(z-1)^2}\right]\Bigg\} \; w = 0.$$

Dividing by $z^{\frac{1}{2}(\alpha_2+\beta_2-5)}(z-1)^{\frac{1}{2}(\alpha_3+\beta_3-5)}$ and simplifying,

$$z^2(z-1)^2\, w'' + \Big\{\tfrac{1}{4}(\alpha_2+\beta_2-1)(\alpha_2+\beta_2-3)(z-1)^2$$

$$+ \tfrac{1}{2}(\alpha_2+\beta_2-1)(\alpha_3+\beta_3-1)z(z-1) + \tfrac{1}{4}(\alpha_3+\beta_3-1)(\alpha_3+\beta_3-3)z^2$$

$$+ \tfrac{1}{2}z(z-1)\Big[(\alpha_2+\beta_2-1)(z-1)+(\alpha_3+\beta_3-1)z\Big]\left(\frac{1-\alpha_2-\beta_2}{z} + \frac{1-\alpha_3-\beta_3}{z-1}\right)$$

$$+ z^2(z-1)^2\left[\frac{\alpha_1\beta_1}{z(z-1)} - \frac{\alpha_2\beta_2}{z^2(z-1)} + \frac{\alpha_3\beta_3}{(z-1)^2}\right]\Big\} \; w = 0.$$

$$z^2(z-1)^2\, w'' + \Big\{\tfrac{1}{4}(\alpha_2+\beta_2-1)(\alpha_2+\beta_2-3)(z-1)^2$$

$$+ \tfrac{1}{2}(\alpha_2+\beta_2-1)(\alpha_3+\beta_3-1)z(z-1) + \tfrac{1}{4}(\alpha_3+\beta_3-1)(\alpha_3+\beta_3-3)z^2$$

$$- \tfrac{1}{2}\Big[(\alpha_2+\beta_2-1)(z-1) + (\alpha_3+\beta_3-1)z\Big]^2 + \alpha_1\beta_1 z(z-1) - \alpha_2\beta_2(z-1)$$

$$+ \alpha_3\beta_3 z^2\Big\} \; w = 0.$$

$$z^2(z-1)^2\,w'' + \Big\{\Big[\tfrac{1}{4}(\alpha_2+\beta_2-1)(\alpha_2+\beta_2-3) + \tfrac{1}{2}(\alpha_2+\beta_2-1)(\alpha_3+\beta_3-1)$$

$$+ \tfrac{1}{4}(\alpha_3+\beta_3-1)(\alpha_3+\beta_3-3) - \tfrac{1}{2}(\alpha_2+\beta_2-1)^2 - (\alpha_2+\beta_2-1)(\alpha_3+\beta_3-1)$$

$$- \tfrac{1}{2}(\alpha_3+\beta_3-1)^2 + \alpha_1\beta_1 + \alpha_3\beta_3\Big]z^2 + \Big[-\tfrac{1}{2}(\alpha_2+\beta_2-1)(\alpha_2+\beta_2-3)$$

$$- \tfrac{1}{2}(\alpha_2+\beta_2-1)(\alpha_3+\beta_3-1) + (\alpha_2+\beta_2-1)^2 + (\alpha_2+\beta_2-1)(\alpha_3+\beta_3-1)$$

$$- \alpha_1\beta_1 - \alpha_2\beta_2\Big]z + \Big[\tfrac{1}{4}(\alpha_2+\beta_2-1)(\alpha_2+\beta_2-3) - \tfrac{1}{2}(\alpha_2+\beta_2-1)^2$$

$$+ \alpha_2\beta_2\Big]\Big\}\ w = 0.$$

Therefore, the normalized form of equation (2-7) is

$$(2\text{-}21)\quad z^2(z-1)^2\,w'' + \Big\{\Big[-\tfrac{1}{4}(\alpha_2+\beta_2)^2 - \tfrac{1}{4}(\alpha_3+\beta_3)^2$$

$$- \tfrac{1}{2}(\alpha_2+\beta_2)(\alpha_3+\beta_3) + \tfrac{1}{2}(\alpha_2+\beta_2) + \tfrac{1}{2}(\alpha_3+\beta_3) + \alpha_1\beta_1$$

$$+ \alpha_3\beta_3\Big]z^2 + \Big[\tfrac{1}{2}(\alpha_2+\beta_2)^2 + \tfrac{1}{2}(\alpha_2+\beta_2)(\alpha_3+\beta_3)$$

$$- \tfrac{1}{2}(\alpha_2+\beta_2) - \tfrac{1}{2}(\alpha_3+\beta_3) - \alpha_1\beta_1 - \alpha_2\beta_2\Big]z$$

$$+ \Big[-\tfrac{1}{4}(\alpha_2+\beta_2)^2 + \tfrac{1}{4} + \alpha_2\beta_2\Big]\Big\}\ w = 0.$$

To obtain the solutions of (2-21) let

$$w = a_o z^{\lambda} + a_1 z^{\lambda+1} + \ldots + a_{n-2} z^{\lambda+n-2} + a_{n-1} z^{\lambda+n-1} + a_n z^{\lambda+n} + \ldots,$$

$$a_o \neq 0,$$

then

$$w'' = a_o\lambda(\lambda-1)z^{\lambda-2}+\ldots+ a_{n-2}(\lambda+n-2)(\lambda+n-3)z^{\lambda+n-4}$$

$$+ a_{n-1}(\lambda+n-1)(\lambda+n-2)z^{\lambda+n-3}+ a_n(\lambda+n)(\lambda+n-1)z^{\lambda+n-2}+\ldots .$$

The indicial equation is

z^{λ}:
$$\lambda(\lambda-1) - \tfrac{1}{4}(\alpha_2+\beta_2)^2 + \tfrac{1}{4} + \alpha_2\beta_2 = 0,$$

(2-22)
$$\lambda^2 - \lambda + \tfrac{1}{4} - \tfrac{1}{4}(\alpha_2-\beta_2)^2 = 0.$$

The exponents are

$$\lambda = \frac{1 \pm \sqrt{1 - 4\left[\frac{1}{4} - \frac{1}{4}(\alpha_2-\beta_2)^2\right]}}{2} = \frac{1 \pm (\alpha_2-\beta_2)}{2} .$$

Finding a_n:

$z^{\lambda+n}$: $a_{n-2}(\lambda+n-2)(\lambda+n-3) - 2a_{n-1}(\lambda+n-1)(\lambda+n-2)$

$$+ a_n(\lambda+n)(\lambda+n-1) + a_{n-2}\left[- \tfrac{1}{4}(\alpha_2+\beta_2)^2 - \tfrac{1}{4}(\alpha_3+\beta_3)^2\right.$$

$$- \tfrac{1}{2}(\alpha_2+\beta_2)(\alpha_3+\beta_3) + \tfrac{1}{2}(\alpha_2+\beta_2) + \tfrac{1}{2}(\alpha_3+\beta_3) + \alpha_1\beta_1$$

$$\left. + \alpha_3\beta_3\right] + a_{n-1}\left[\tfrac{1}{2}(\alpha_2+\beta_2)^2 + \tfrac{1}{2}(\alpha_2+\beta_2)(\alpha_3+\beta_3)\right.$$

$$\left. - \tfrac{1}{2}(\alpha_2+\beta_2) - \tfrac{1}{2}(\alpha_3+\beta_3) - \alpha_1\beta_1 - \alpha_2\beta_2\right] + a_n\left[- \tfrac{1}{4}(\alpha_2+\beta_2)^2\right.$$

$$\left. + \tfrac{1}{4} + \alpha_2\beta_2\right] = 0.$$

$$(2\text{-}23)\quad a_n = \Big\{ a_{n-2}\Big[-(\lambda+n-2)(\lambda+n-3) + \tfrac{1}{4}(\alpha_2+\beta_2)^2 + \tfrac{1}{4}(\alpha_3+\beta_3)^2$$

$$+ \tfrac{1}{2}(\alpha_2+\beta_2)(\alpha_3+\beta_3) - \tfrac{1}{2}(\alpha_2+\beta_2) - \tfrac{1}{2}(\alpha_3+\beta_3) - \alpha_1\beta_1$$

$$- \alpha_3\beta_3 \Big] + a_{n-1}\Big[2(\lambda+n-1)(\lambda+n-2) - \tfrac{1}{2}(\alpha_2+\beta_2)^2$$

$$- \tfrac{1}{2}(\alpha_2+\beta_2)(\alpha_3+\beta_3) + \tfrac{1}{2}(\alpha_2+\beta_2) + \tfrac{1}{2}(\alpha_3+\beta_3)$$

$$+ \alpha_1\beta_1 + \alpha_2\beta_2 \Big]\Big\} \frac{1}{(\lambda+n)(\lambda+n-1) - \tfrac{1}{4}(\alpha_2-\beta_2)^2 + \tfrac{1}{4}} .$$

In order to simplify further work we can, without loss of generality, let

$$\tfrac{1}{4}(\alpha_2+\beta_2)^2 + \tfrac{1}{4}(\alpha_3+\beta_3)^2 + \tfrac{1}{2}(\alpha_2+\beta_2)(\alpha_3+\beta_3) - \tfrac{1}{2}(\alpha_2+\beta_2)$$

$$- \tfrac{1}{2}(\alpha_3+\beta_3) - \alpha_1\beta_1 - \alpha_3\beta_3 = k$$

and

$$- \tfrac{1}{2}(\alpha_2+\beta_2)^2 - \tfrac{1}{2}(\alpha_2+\beta_2)(\alpha_3+\beta_3) + \tfrac{1}{2}(\alpha_2+\beta_2) + \tfrac{1}{2}(\alpha_3+\beta_3) + \alpha_1\beta_1$$

$$+ \alpha_2\beta_2 = m.$$

Making these substitutions in (2-23) and simplifying the denominator by use of the indicial equation (2-22), we have

$$a_n = \frac{a_{n-2}\Big[-(\lambda+n-2)(\lambda+n-3)+k\Big] + a_{n-1}\Big[2(\lambda+n-1)(\lambda+n-2) + m\Big]}{n(2\lambda+n-1)} .$$

$$a_1 = \frac{a_o\left[2\lambda(\lambda-1) + m\right]}{1(2\lambda)} = \frac{a_o(2\lambda^2-2\lambda+m)}{1!(2\lambda)} .$$

$$a_2 = \frac{a_o\left[-\lambda(\lambda-1) + k\right] + a_1\left[2(\lambda+1)\lambda + m\right]}{2(2\lambda+1)} ,$$

$$a_2 = a_o\left[\frac{2\lambda(-\lambda^2+\lambda+k) + (2\lambda^2-2\lambda+m)(2\lambda^2+2\lambda+m)}{2!\ 2\lambda(2\lambda+1)} \right] ,$$

$$a_2 = a_o\left[\frac{4\lambda^4-2\lambda^3 + (4m-2)\lambda^2 + 2k\lambda + m^2}{2!\ (2\lambda)(2\lambda+1)} \right] .$$

$$a_3 = \frac{a_1\left[-(\lambda+1)\lambda + k\right] + a_2\left[2(\lambda+2)(\lambda+1) + m\right]}{3(2\lambda+2)} ,$$

$$a_3 = a_o \left\{ 2(-\lambda^2-\lambda+k)(2\lambda^2-2\lambda+m)(2\lambda+1)+(2\lambda^2+6\lambda+4+m)\left[4\lambda^4-2\lambda^3 \right.\right.$$

$$\left.\left. + (4m-2)\lambda^2 + 2k\lambda + m^2\right] \right\} \frac{1}{3!\ (2\lambda)(2\lambda+1)(2\lambda+2)} ,$$

$$a_3 = a_o\Big[8\lambda^6 + 12\lambda^5 + 4(3m-1)\lambda^4 + 6(3m+2k-2)\lambda^3$$

$$+ 2(3m^2+4m-2+4k)\lambda^2 + 2(3m^2-m+3mk+2k)\lambda$$

$$+ m(m^2+4m+2k)\Big] \frac{1}{3!\ (2\lambda)(2\lambda+1)(2\lambda+2)} .$$

The solutions of equation (2-21) are

$$(2\text{-}24)\qquad w = a_o z^{\lambda}\left[1 + \frac{(2\lambda^2-2\lambda+m)}{1!(2\lambda)}\, z + \frac{4\lambda^4-2\lambda^3+(4m-2)\lambda^2 + 2k\lambda + m^2}{2!(2\lambda)(2\lambda+1)}\, z^2 + \ldots\right],$$

where $\lambda = \frac{1\pm(\alpha_2-\beta_2)}{2}$ and k and m have the values:

$$k = \tfrac{1}{4}(\alpha_2+\beta_2+\alpha_3+\beta_3)^2 - \tfrac{1}{2}(\alpha_2+\beta_2+\alpha_3+\beta_3) - \alpha_1\beta_1 - \alpha_3\beta_3,$$

$$m = -\tfrac{1}{2}(\alpha_2+\beta_2-1)(\alpha_2+\beta_2+\alpha_3+\beta_3) + \alpha_1\beta_1 + \alpha_2\beta_2.$$

2.5 Factored Solution $W_{\alpha,k,m}$ and Hypergeometric Solution $W_{\alpha,k}$. Next, we shall consider solutions of the equation

$$(2\text{-}25)\qquad u = \left\{\begin{array}{cccc} \infty & 0 & 1 & \infty \\ \alpha_1 & \alpha & 2k & \alpha_4 \\ \frac{m}{\alpha_1} & 1-\alpha & 0 & \beta_4 \end{array}\; z\right\}$$

when the u' term has been removed. This can be obtained from equation (2-21) by letting $\alpha_1\beta_1 = m$, $\alpha_2 = \alpha$, $\beta_2 = 1-\alpha$, $\alpha_3 = 2k$, $\beta_3=0$. Although $\beta_3=0$ in the normalized equation, this does not insure a two term recurrence relation as it would in the original equation (2-7). Our transformed equation, that is (2-25) in normalized form, is

$$(2\text{-}26)\quad z^2(z-1)^2w'' + \left[(m+\tfrac{1}{4}-k^2)z^2+(\alpha^2-\alpha-m)z+\alpha(1-\alpha)\right]w = 0.$$

Assume the solution

$$w = C_o z^{\lambda}+\ldots+ C_{n-2}z^{\lambda+n-2}+ C_{n-1}z^{\lambda+n-1}+ C_n z^{\lambda+n}+\ldots, \quad C_o \neq 0,$$

then

$$w'' = C_o\lambda(\lambda-1)z^{\lambda-2} +\ldots+ C_{n-2}(\lambda+n-2)(\lambda+n-3)z^{\lambda+n-4}$$

$$+ C_{n-1}(\lambda+n-1)(\lambda+n-2)z^{\lambda+n-3}+ C_n(\lambda+n)(\lambda+n-1)z^{\lambda+n-2} +\ldots.$$

The indicial equation and exponents are

z^{λ}:

$$\lambda(\lambda-1) + \alpha - \alpha^2 = 0,$$

$$\lambda^2 - \lambda + \alpha(1-\alpha) = 0,$$

$$(\lambda-\alpha)(\lambda+\alpha-1) = 0,$$

$$\lambda = \alpha,\ 1-\alpha.$$

Obtaining C_n:

$z^{\lambda+n}$:

$$C_{n-2}(\lambda+n-2)(\lambda+n-3)-2C_{n-1}(\lambda+n-1)(\lambda+n-2)$$

$$+ C_n(\lambda+n)(\lambda+n-1) + C_{n-2}(m+\tfrac{1}{4}-k^2)$$

$$+ C_{n-1}(\alpha^2-\alpha-m) + C_n(\alpha-\alpha^2) = 0.$$

$$C_n = \left\{ -C_{n-2}\left[(\lambda+n-2)(\lambda+n-3)+(m+\tfrac{1}{4}-k^2) \right]+C_{n-1}\left[2(\lambda+n-1)(\lambda+n-2) \right.\right.$$

$$\left.\left. - (\alpha^2 - \alpha - m) \right] \right\} \frac{1}{(\lambda+n)(\lambda+n-1) + \alpha - \alpha^2} \quad .$$

Let $\lambda = \alpha$:

$$C_n = \left\{ C_{n-2}\left[-(\alpha+n-2)(\alpha+n-3)-m-\tfrac{1}{4}+k^2 \right] + C_{n-1}\left[\alpha^2 + (4n-5)\alpha \right.\right.$$

$$\left.\left. +2(n-1)(n-2) + m \right] \right\} \frac{1}{n(2\alpha+n-1)} \quad .$$

$$C_1 = \frac{C_o\left[\alpha^2-\alpha+m \right]}{1(2\alpha)} \quad .$$

$$C_2 = \frac{C_o\left[-\alpha(\alpha-1)-m-\tfrac{1}{4}+k^2 \right] + C_1\left[\alpha^2+3\alpha+m \right]}{2(2\alpha+1)} \quad ,$$

$$C_2 = C_o\left[\frac{2\alpha(-\alpha^2+\alpha-m-\tfrac{1}{4}+k^2)+(\alpha^2-\alpha+m)(\alpha^2+3\alpha+m)}{2!(2\alpha)(2\alpha+1)} \right] ,$$

$$C_2 = C_o\left[\frac{\alpha^4+(2m-1)\alpha^2 + (2k^2-\tfrac{1}{2})\alpha + m^2}{2!(2\alpha)(2\alpha+1)} \right] .$$

$$C_3 = \frac{C_1\left[-\alpha(\alpha+1)-m-\tfrac{1}{4}+k^2 \right] + C_2\left[\alpha^2+7\alpha+4+m \right]}{3(2\alpha+2)} \quad ,$$

$$C_3 = C_o\left\{2(\alpha^2-\alpha+m)(-\alpha^2-\alpha-m-\tfrac{1}{4}+k^2)(2\alpha+1)+\left[\alpha^4+(2m-1)\alpha^2\right.\right.$$

$$\left.\left.+(2k^2-\tfrac{1}{2})\alpha+m^2\right](\alpha^2+7\alpha+4+m)\right\}\frac{1}{3!(2\alpha)(2\alpha+1)(2\alpha+2)},$$

$$C_3 = C_o\left[\alpha^6+3\alpha^5+(3m+1)\alpha^4+(6m+6k^2-\tfrac{9}{2})\alpha^3+(12k^2+3m+3m^2-5)\alpha^2\right.$$

$$\left.+(3m^2+6k^2+6mk^2-\tfrac{3}{2}m-\tfrac{3}{2})\alpha+(m^3+2m^2+2k^2m-\tfrac{1}{2}m)\right]$$

$$\cdot\frac{1}{3!2\alpha(2\alpha+1)(2\alpha+2)}.$$

Thus one solution of (2-26) can be written with a few terms as:

$$(2\text{-}27)\quad w = C_o z^{\alpha}\left[1+\frac{\alpha^2-\alpha+m}{1!\ 2\alpha}z+\frac{\alpha^4+(2m-1)\alpha^2+(2k^2-\frac{1}{2})\alpha+m^2}{2!\ 2\alpha(2\alpha+1)}z^2+\ldots\right].$$

A second solution of (2-26) can be obtained by replacing α by $1-\alpha$:

$$(2\text{-}28)\quad w = C_o z^{1-\alpha}\left[1+\frac{\alpha^2-\alpha+m}{1!(2-2\alpha)}z\right.$$

$$\left.+\frac{\alpha^4-4\alpha^3+(5+2m)\alpha^2+(-2k^2-4m-\frac{3}{2})\alpha+(m^2+2m-\frac{1}{2}+2k^2)}{2!\ (2-2\alpha)(3-2\alpha)}z^2+\ldots\right].$$

Although solution (2-27) has no obvious general term, it can be shown by a lengthy process of long division (which is omitted here), that it can be written in the factored form:

$$w=C_o z^{\alpha}(1-z)^{k+\frac{1}{2}}\left\{1+\frac{\alpha^2+2k\alpha+m}{1!\ 2\alpha}z+\frac{(\alpha^2+2k\alpha+m)\left[\alpha^2+(2k+2)\alpha+2k+m+1\right]}{2!\ 2\alpha(2\alpha+1)}z^2\right.$$

$$+\frac{(\alpha^2+2k\alpha+m)\left[\alpha^2+(2k+2)\alpha+2k+m+1\right]\left[\alpha^2+(2k+4)\alpha+4k+m+4\right]}{3!\ 2\alpha(2\alpha+1)(2\alpha+2)}z^3+\ldots$$

$$+\left((\alpha^2+2k\alpha+m)\left[\alpha^2+(2k+2)\alpha+2k+m+1\right]\cdots\left[\alpha^2+2(k+n-1)\alpha+m\right.\right.$$

$$\left.\left.+(n-1)(2k+n-1)\right]\right)\cdot\frac{z^n}{n!\ 2\alpha(2\alpha+1)\ldots(2\alpha+n-1)}+\ldots\Bigg\}.$$

Let this solution be indicated $W_{\alpha,k,m}$, since α,k,m are the only parameters; therefore (2-27) can be written

$$(2\text{-}29)\qquad W_{\alpha,k,m}=C_o z^{\alpha}(1-z)^{k+\frac{1}{2}}\left\{1+\sum_{n=1}^{\infty}(\alpha^2+2k\alpha+m)\right.$$

$$\left.\cdot\frac{\left[\alpha^2+2(k+1)\alpha+2k+1+m\right]\cdots\left[\alpha^2+2(k+n-1)\alpha+(n-1)(2k+n-1)+m\right]}{n!\ 2\alpha(2\alpha+1)\cdots(2\alpha+n-1)}z^n\right\}.$$

If $m=0$, then the solution (2-29) becomes hypergeometric. Let this solution be indicated $W_{\alpha,k}$, α and k being the only parameters:

$$W_{\alpha,k}=C_o z^{\alpha}(1-z)^{k+\frac{1}{2}}\left\{1\right.$$

$$\left.+\sum_{n=1}^{\infty}\frac{(\alpha^2+2k\alpha)\left[\alpha^2+2(k+1)\alpha+2k+1\right]\cdots\left[\alpha^2+2(k+n-1)\alpha+(n-1)(2k+n-1)\right]}{n!\ 2\alpha(2\alpha+1)\ldots(2\alpha+n-1)}z^n\right\}.$$

Factoring further we may write

$$W_{\alpha,k} = C_o z^{\alpha}(1-z)^{k+\frac{1}{2}} \left\{ 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+2k)\left[(\alpha+1)(\alpha+2k+1)\right]\cdots\left[(\alpha+n-1)(\alpha+2k+n-1)\right]}{n!\ 2\alpha(2\alpha+1)\cdots(2\alpha+n-1)} z^n \right\}.$$

Rearranging factors we arrive at the solution

$$W_{\alpha,k} = C_o z^{\alpha}(1-z)^{k+\frac{1}{2}} \left\{ 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)(\alpha+2k)(\alpha+2k+1)\cdots(\alpha+2k+n-1)}{n!\ 2\alpha(2\alpha+1)\cdots(2\alpha+n-1)} z^n \right\},$$

which is hypergeometric in form (2-18) and shall be denoted by $W_{\alpha,k}$. Therefore, when $m = 0$ in (2-26) we have the solution

$$W_{\alpha,k} = z^{\alpha}(1-z)^{k+\frac{1}{2}}\ {}_2F_1(\alpha,\alpha+2k;\ 2\alpha;\ z), \tag{2-30}$$

where we have let $C_o = 1$. A second solution can be obtained by replacing α by $1-\alpha$.

2.6 Notation and Proofs Involving ${}_2F_1(a, b; c; z)$.

Before presenting several interesting properties of $W_{\alpha,k}$, we shall introduce notation and prove seven statements that will be useful in section 2.7.

Notation:

n, j are integers.

$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$.

$$\binom{n}{j} = \frac{n(n-1)\cdots(n-j+1)}{j!} \; ; \quad \binom{n}{0} = 1.$$

$$D^n W = \frac{d^n W}{dz^n} .$$

$$W_{\alpha,k} = W_{\alpha,k}(z).$$

If $F = {}_2F_1(a, b; c; z) = F(a, b; c; z)$, then

$$F(a+) = F(a+1, b; c; z), \quad [5, p.50] ,$$

$$F(b+) = F(a, b+1; c; z),$$

$$F(c+) = F(a, b; c+1; z),$$

$$F(a-) = F(a-1, b; c; z), \text{ etc.},$$

$$F(+n) = F(a+n, b+n; c+n; z), \quad n = 1,2,3,\ldots,$$

$$F(-1) = F(a-1, b-1; c-1; z).$$

We shall prove the following:

(2-31) $$F = \frac{1}{a-c+1}\left[aF(a+) + (1-c)F(c-)\right],$$

(2-32) $$F = \frac{1}{1-z}\left[F(a-) + \frac{(b-c)z}{c} F(c+)\right],$$

(2-33) $$F(+1) = \frac{c}{az}\left[F(b+) - F\right],$$

(2-34) $$D^n F = \frac{(a)_n (b)_n}{(c)_n} F(+n);$$

and for $F = {}_2F_1(\alpha, \alpha+2k; 2\alpha; z) = F(\alpha, \alpha+2k; 2\alpha; z)$,

(2-35) $F(+n)$

$$= \frac{(2\alpha)_n}{(\alpha)_n} z^{-\alpha-n}(1-z)^{-k-\frac{n+1}{2}} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (1-z)^{\frac{j}{2}} W_{\alpha,k+\frac{n-j}{2}} ,$$

$$(2\text{-}36)\quad D^n\left[z^{\alpha}(1-z)^{k+\frac{1}{2}}\right]$$

$$= z^{\alpha}(1-z)^{k-n+\frac{1}{2}} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (-\alpha)_j \left(-k-\tfrac{1}{2}\right)_{n-j} z^{-j} (1-z)^j ,$$

$$(2\text{-}37)\quad F(-1) = \frac{(1-z)^{-k} z^{1-\alpha}}{(2\alpha-1)(2-z)}\left[2(2\alpha-1)(1-z)^{\frac{1}{2}} W_{\alpha-1,k} + (\alpha-2k) W_{\alpha,k-\frac{1}{2}}\right].$$

Finally, we state Leibniz's rule [9, p. 409] for finding the nth derivative but shall not prove it:

$$(2\text{-}38)\quad D^n(uv) = u\, D^n v + \binom{n}{1} Du\, D^{n-1} v + \binom{n}{2} D^2 u\, D^{n-2} v + \ldots$$

$$+ \binom{n}{\ell} D^{\ell} u\, D^{n-\ell} v + \ldots + (D^n u) v$$

$$= \sum_{\ell=0}^{n} \binom{n}{\ell} D^{\ell} u\, D^{n-\ell} v .$$

PROOFS

$$(2\text{-}31)\qquad F = \frac{1}{a-c+1}\left[aF(a+) + (1-c)F(c-)\right] .$$

Proof:

$$aF(a+) + (1-c)F(c-)$$

$$= a\sum_{n=0}^{\infty} \frac{(a+1)_n (b)_n}{(c)_n n!} z^n + (1-c) \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c-1)_n n!} z^n$$

$$= \sum_{n=0}^{\infty} \left[\frac{a(a+1)_n (b)_n}{(c)_n n!} + (1-c)\, \frac{(a)_n (b)_n}{(c-1)_n n!}\right] z^n$$

$$= \sum_{n=0}^{\infty} \left[a+n-(c+n-1) \right] \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

$$= (a - c + 1)F.$$

Therefore,

$$F = \frac{1}{a-c+1} \left[aF(a+) + (1-c)F(c-) \right] .$$

(2-32) $$F = \frac{1}{1-z} \left[F(a-) + \frac{(b-c)z}{c} F(c+) \right] .$$

Proof:

$$F(a-) + \frac{(b-c)z}{c} F(c+)$$

$$= \sum_{n=0}^{\infty} \frac{(a-1)_n (b)_n}{(c)_n n!} z^n + \frac{(b-c)z}{c} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c+1)_n n!} z^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(a-1)_n (b)_n}{(c)_n n!} z^n + \frac{(b-c)}{c} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c+1)_n n!} z^{n+1}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(a-1)_n (b)_n}{(c)_n n!} z^n + \frac{(b-c)}{c} \sum_{n=1}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c+1)_{n-1} (n-1)!} z^n$$

$$= 1 + \sum_{n=1}^{\infty} \left[\frac{(a-1)}{a+n-1} + \frac{n(b-c)}{(a+n-1)(b+n-1)} \right] \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

$$= 1 + \sum_{n=1}^{\infty} \left[\frac{(a-1)(b+n-1)+n(b-c)}{(a+n-1)(b+n-1)} \right] \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

$$= 1 + \sum_{n=1}^{\infty} \left[1 - \frac{n(c+n-1)}{(a+n-1)(b+n-1)} \right] \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

$$= F - \sum_{n=1}^{\infty} \left[\frac{n(c+n-1)}{(a+n-1)(b+n-1)} \right] \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

$$= F - \sum_{n=1}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (n-1)!} z^n$$

$$= F - \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^{n+1}$$

$$= F - zF$$

$$= (1-z)F.$$

Thus,

$$F(a-) + \frac{(b-c)z}{c} F(c+) = (1-z)F$$

or

$$F = \frac{1}{1-z} \left[F(a-) + \frac{(b-c)z}{c} F(c+) \right].$$

(2-33)
$$F(+1) = \frac{c}{az} \left[F(b+) - F \right].$$

Proof:

$$F(b+) - F = \sum_{n=0}^{\infty} \frac{(a)_n (b+1)_n}{(c)_n n!} z^n - \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

$$= \sum_{n=1}^{\infty} \left[\frac{b+n}{n} - \frac{b}{n} \right] \frac{(a)_n (b+1)_{n-1}}{(c)_n (n-1)!} z^n$$

$$= \sum_{n=1}^{\infty} \frac{(a)_n (b+1)_{n-1}}{(c)_n (n-1)!} z^n$$

$$= \frac{az}{c} \sum_{n=1}^{\infty} \frac{(a+1)_{n-1} (b+1)_{n-1}}{(c+1)_{n-1} (n-1)!} z^{n-1}$$

$$= \frac{az}{c} \sum_{n=0}^{\infty} \frac{(a+1)_n (b+1)_n}{(c+1)_n n!} z^n .$$

Hence,

$$F(b+) - F = \frac{az}{c} F(+1)$$

or

$$F(+1) = \frac{c}{az} \left[F(b+) - F \right] .$$

(2-34) $$D^n F = \frac{(a)_n (b)_n}{(c)_n} F(+n) .$$

Proof:

$$D \left[F(a, b; c; z) \right] = D \left[\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \right]$$

$$= \sum_{n=1}^{\infty} \frac{n(a)_n (b)_n}{(c)_n n!} z^{n-1}$$

$$= \sum_{n=0}^{\infty} \frac{(n+1)(a)_{n+1} (b)_{n+1}}{(c)_{n+1} (n+1)!} z^n$$

$$= \frac{ab}{c} \sum_{n=0}^{\infty} \frac{(a+1)_n (b+1)_n}{(c+1)_n n!} z^n$$

$$= \frac{ab}{c} F(+1).$$

Similarly,

$$D^2 \left[F(a, b; c; z) \right] = \frac{a(a+1)b(b+1)}{c(c+1)} \sum_{n=1}^{\infty} \frac{(a+2)_{n-1} (b+2)_{n-1}}{(c+2)_{n-1} (n-1)!} z^{n-1}$$

$$= \frac{(a)_2 (b)_2}{(c)_2} \sum_{n=0}^{\infty} \frac{(a+2)_n (b+2)_n}{(c+2)_n n!} z^n$$

$$= \frac{(a)_2 (b)_2}{(c)_2} F(+2).$$

In general,

$$D^n \left[F(a, b; c; z) \right] = \frac{(a)_n (b)_n}{(c)_n} F(+n), \quad n=1,2,3,\ldots.$$

If $F = F(\alpha, \alpha+2k; 2\alpha; z) = z^{-\alpha}(1-z)^{-k-\frac{1}{2}} W_{\alpha,k}$,

then

(2-35) $F(+n)$

$$= \frac{(2\alpha)_n}{(\alpha)_n} z^{-\alpha-n}(1-z)^{-k-\frac{n+1}{2}} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (1-z)^{\frac{j}{2}} W_{\alpha,k+\frac{n-j}{2}} .$$

Note that when k is replaced by k + m in the premise, then

$$F(\alpha, \alpha+2k+2m; 2\alpha; z) = z^{-\alpha}(1-z)^{-k-m-\frac{1}{2}} W_{\alpha,k+m} .$$

Proof:

We begin by applying (2-33) to F(+1) and express the result as a function of $W_{\alpha,k+m}$:

$$F(+1) = \frac{2\alpha}{\alpha z}\left[F(\alpha,\ \alpha+2k+1;\ 2\alpha;\ z) - F\right]$$

$$= \frac{2\alpha}{\alpha z}\left[z^{-\alpha}(1-z)^{-k-1}W_{\alpha,k+\frac{1}{2}} - z^{-\alpha}(1-z)^{-k-\frac{1}{2}}W_{\alpha,k}\right].$$

$$(2\text{-}39)\qquad F(+1) = \frac{(2\alpha)}{(\alpha)}\, z^{-\alpha-1}(1-z)^{-k-1}\left[W_{\alpha,k+\frac{1}{2}} - (1-z)^{\frac{1}{2}}W_{\alpha,k}\right].$$

Apply the principle of (2-33) to F(+2):

$$(2\text{-}40)\qquad F(+2) = \frac{(2\alpha+1)}{(\alpha+1)z}\left[F(\alpha+1,\ \alpha+2k+2;\ 2\alpha+1;\ z) - F(+1)\right].$$

Using the expression (2-39) with k replaced by $k + \frac{1}{2}$ in (2-40), we have

$$(2\text{-}41)\quad F(+2)$$

$$= \frac{(2\alpha+1)}{(\alpha+1)z}\left\{\frac{2\alpha}{\alpha}\, z^{-\alpha-1}(1-z)^{-k-\frac{3}{2}}\left[W_{\alpha,k+1} - (1-z)^{\frac{1}{2}}W_{\alpha,k+\frac{1}{2}}\right]\right.$$

$$\left. - \frac{2\alpha}{\alpha}\, z^{-\alpha-1}(1-z)^{-k-1}\left[W_{\alpha,k+\frac{1}{2}} - (1-z)^{\frac{1}{2}}W_{\alpha,k}\right]\right\}$$

$$= \frac{(2\alpha)_2}{(\alpha)_2}\, z^{-\alpha-2}(1-z)^{-k-\frac{3}{2}}\left[W_{\alpha,k+1} - 2(1-z)^{\frac{1}{2}}W_{\alpha,k+\frac{1}{2}}\right.$$

$$\left. + (1-z)W_{\alpha,k}\right].$$

Let us find F(+3) before generalizing our results to F(+n). Upon applying (2-33) we have

$$(2\text{-}42) \qquad F(+3) = \frac{(2\alpha+2)}{(\alpha+2)z}\left[F(\alpha+2,\ \alpha+2k+3;\ 2\alpha+2;\ z) - F(+2)\right] .$$

Replace k by $k + \frac{1}{2}$ in (2-41) to write (2-42) in the form

$$F(+3) = \frac{(2\alpha+2)}{(\alpha+2)z}\left\{\frac{(2\alpha)_2}{(\alpha)_2} z^{-\alpha-2}(1-z)^{-k-2}\left[W_{\alpha,k+\frac{3}{2}} - 2(1-z)^{\frac{1}{2}}W_{\alpha,k+1}\right.\right.$$

$$\left. + (1-z)W_{\alpha,k+\frac{1}{2}}\right] - \frac{(2\alpha)_2}{(\alpha)_2} z^{-\alpha-2}(1-z)^{-k-\frac{3}{2}}\left[W_{\alpha,k+1}\right.$$

$$\left.\left. - 2(1-z)^{\frac{1}{2}}W_{\alpha,k+\frac{1}{2}} + (1-z)\ W_{\alpha,k}\right]\right\} .$$

Simplifying further

$$F(+3) = \frac{(2\alpha)_3}{(\alpha)_3} z^{-\alpha-3}(1-z)^{-k-2}\left[W_{\alpha,k+\frac{3}{2}} - 3(1-z)^{\frac{1}{2}}W_{\alpha,k+1}\right.$$

$$\left. + 3(1-z)W_{\alpha,k+\frac{1}{2}} - (1-z)W_{\alpha,k}\right] .$$

Finally,

$$F(+n) = \frac{(2\alpha)_n}{(\alpha)_n} z^{-\alpha-n}(1-z)^{-k-\frac{n+1}{2}}\left[W_{\alpha,k+\frac{n}{2}} - \binom{n}{1}(1-z)^{\frac{1}{2}}W_{\alpha,k+\frac{n-1}{2}}\right.$$

$$+ \binom{n}{2}(1-z)W_{\alpha,k+\frac{n-2}{2}} + \ldots + (-1)^j\binom{n}{j}(1-z)^{\frac{j}{2}}W_{\alpha,k+\frac{n-j}{2}}$$

$$\left. + \ldots + (1-z)^{\frac{n}{2}}W_{\alpha,k}\right] .$$

Therefore,

$$F(+n) = \frac{(2\alpha)_n}{(\alpha)_n} z^{-\alpha-n}(1-z)^{-k-\frac{n+1}{2}} \sum_{j=0}^{n} (-1)^j\binom{n}{j}(1-z)^{\frac{j}{2}}W_{\alpha,k+\frac{n-j}{2}} ,$$

$$n=1,2,3,\ldots .$$

(2-36) $$D^n\left[z^\alpha(1-z)^{k+\frac{1}{2}}\right]$$

$$= z^\alpha(1-z)^{k-n+\frac{1}{2}}\sum_{j=0}^{n}(-1)^j\binom{n}{j}(-\alpha)_j(-k-\tfrac{1}{2})_{n-j}z^{-j}(1-z)^j \ .$$

Using Leibniz's rule where $u = z^\alpha$ and $v = (1-z)^{k+\frac{1}{2}}$, find $D^\ell u$ and $D^{n-\ell}v$. When $u = z^\alpha$,

$$Du = \alpha z^{\alpha-1}; \quad D^2u = \alpha(\alpha-1)z^{\alpha-2}; \ldots; \quad D^\ell u = (-1)^\ell(-\alpha)_\ell z^{\alpha-\ell} \ .$$

If $v = (1-z)^{k+\frac{1}{2}}$, then

$$Dv = -(k+\tfrac{1}{2})(1-z)^{k-\frac{1}{2}}; \quad D^2v = (k+\tfrac{1}{2})(k-\tfrac{1}{2})(1-z)^{k-\frac{3}{2}}; \ldots;$$

$$D^{n-\ell}v = (-k-\tfrac{1}{2})_{n-\ell}(1-z)^{k+\ell-n+\frac{1}{2}} \ .$$

Hence, the nth derivative is found to be

$$D^n\left[z^\alpha(1-z)^{k+\frac{1}{2}}\right] = \sum_{\ell=0}^{n}\binom{n}{\ell}(-1)^\ell(-\alpha)_\ell z^{\alpha-\ell}(-k-\tfrac{1}{2})_{n-\ell}(1-z)^{k+\ell-n+\frac{1}{2}}$$

$$= z^\alpha(1-z)^{k-n+\frac{1}{2}}\sum_{\ell=0}^{n}(-1)^\ell\binom{n}{\ell}(-\alpha)_\ell(-k-\tfrac{1}{2})_{n-\ell}z^{-\ell}(1-z)^\ell \ .$$

If $F(-1) = F(\alpha-1, \alpha+2k-1; 2\alpha-1; z)$ show that

(2-37)
$$F(-1) = \frac{(1-z)^{-k}z^{1-\alpha}}{(2\alpha-1)(2-z)}\left[2(2\alpha-1)(1-z)^{\frac{1}{2}}W_{\alpha-1,k}+(\alpha-2k)W_{\alpha,k-\frac{1}{2}}\right] \ .$$

Apply (2-31) to F(-1):

$F(\alpha-1, \alpha+2k-1; 2\alpha-1; z)$

$$= \frac{1}{-\alpha+1}\left[(\alpha-1)F(\alpha,\ \alpha+2k-1;\ 2\alpha-1;\ z)\right.$$

$$\left. + (2-2\alpha)F(\alpha-1,\ \alpha+2k-1;\ 2\alpha-2;\ z)\right]$$

$$= -F(\alpha,\ \alpha+2k-1;\ 2\alpha-1;\ z) + 2F(\alpha-1,\ \alpha+2k-1;\ 2\alpha-2;\ z).$$

Now apply (2-32) to the first term with the result that

$$F(-1) = \frac{-1}{1-z}\left[F(-1) + \frac{(2k-\alpha)z}{2\alpha-1}F(\alpha,\ \alpha+2k-1;\ 2\alpha;\ z)\right]$$

$$+ 2F(\alpha-1,\ \alpha+2k-1;\ 2\alpha-2;\ z).$$

Collecting the terms containing F(-1) we have

$$\text{(2-38)}\quad \left(1 + \frac{1}{1-z}\right)F(-1) = \frac{\alpha-2k}{2\alpha-1}\cdot\frac{z}{1-z}F(\alpha,\ \alpha+2k-1;\ 2\alpha;\ z)$$

$$+ 2F(\alpha-1,\ \alpha+2k-1;\ 2\alpha-2;\ z).$$

Now, since

$$\text{(2-39)}\qquad F(\alpha,\ \alpha+2k;\ 2\alpha;\ z) = z^{-\alpha}(1-z)^{-k-\frac{1}{2}}W_{\alpha,k}\ ,$$

if k is replaced by $k - \frac{1}{2}$, then

$$F(\alpha,\ \alpha+2k-1;\ 2\alpha;\ z) = z^{-\alpha}(1-z)^{-k}W_{\alpha,k-\frac{1}{2}}.$$

If α is replaced by $\alpha-1$ in (2-39), then

$$F(\alpha-1,\ \alpha+2k-1;\ 2\alpha-2;\ z) = z^{-\alpha+1}(1-z)^{-k-\frac{1}{2}}W_{\alpha-1,k}\ .$$

Using these two results and solving equation (2-38) for F(-1), permits the result

$$F(-1) = \frac{(1-z)}{(2-z)}\left[\frac{\alpha-2k}{2\alpha-1}\cdot\frac{z}{1-z}\,z^{-\alpha}(1-z)^{-k}W_{\alpha,k-\frac{1}{2}}\right.$$

$$\left. + 2z^{-\alpha+1}(1-z)^{-k-\frac{1}{2}}W_{\alpha-1,k}\right]$$

$$= \frac{z^{-\alpha+1}(1-z)^{-k}}{(2-z)}\left[\frac{(\alpha-2k)}{2\alpha-1}\,W_{\alpha,k-\frac{1}{2}} + 2(1-z)^{\frac{1}{2}}W_{\alpha-1,k}\right],$$

or

$$F(-1) = \frac{(1-z)^{-k}z^{1-\alpha}}{(2\alpha-1)(2-z)}\left[(\alpha-2k)W_{\alpha,k-\frac{1}{2}} + 2(2\alpha-1)(1-z)^{\frac{1}{2}}W_{\alpha-1,k}\right].$$

2.7 Recurrence Relations, nth Derivative, Sum and Product Formulas for $W_{\alpha,k}$. By utilizing the statements we have just verified, it is possible to obtain recurrence relations for $W_{\alpha,k}$. Several such relations will now be presented.

By replacing α by $\alpha-1$ in equation (2-30), we may write

$$(2\text{-}40)\quad W_{\alpha-1,k} = z^{\alpha-1}(1-z)^{k+\frac{1}{2}}F(\alpha-1,\ \alpha+2k-1;\ 2\alpha-2;\ z).$$

Since the parameters a and b in F(a, b; c; z) can be interchanged without any lose in generality, we can also interchange a and b in equation (2-32) to obtain the valid relationship

$$F = \frac{1}{1-z}\left[F(b-) + \frac{(a-c)z}{c}\,F(c+)\right].$$

Now, if this be applied to (2-40), we may conclude that

$$(2\text{-}41)\quad W_{\alpha-1,k} = z^{\alpha-1}(1-z)^{k-\frac{1}{2}}\Big[F(\alpha-1,\ \alpha+2k-2;\ 2\alpha-2;\ z)$$

$$+ \frac{(1-\alpha)z}{2\alpha-2}\,F(-1)\Big]\,.$$

Replacing α by $\alpha - 1$ and k by $k - \frac{1}{2}$ in (2-30), it is seen that

$$F(\alpha-1,\ \alpha+2k-2;\ 2\alpha-2;\ z) = z^{1-\alpha}(1-z)^{-k}W_{\alpha-1,k-\frac{1}{2}}.$$

This, together with (2-37) enables us to write (2-41) in the form

$$W_{\alpha-1,k} = z^{\alpha-1}(1-z)^{k-\frac{1}{2}}\Big\{z^{1-\alpha}(1-z)^{-k}W_{\alpha-1,k-\frac{1}{2}}$$

$$- \frac{1}{2(2\alpha-1)(2-z)}\,(1-z)^{-k}z^{2-\alpha}\Big[2(2\alpha-1)(1-z)^{\frac{1}{2}}W_{\alpha-1,k}$$

$$+ (\alpha-2k)W_{\alpha,k-\frac{1}{2}}\Big]\Big\}$$

$$= (1-z)^{-\frac{1}{2}}W_{\alpha-1,k-\frac{1}{2}} - \frac{z}{(2-z)}\,W_{\alpha-1,k}$$

$$- \frac{(\alpha-2k)}{2(2\alpha-1)(2-z)}\,z(1-z)^{-\frac{1}{2}}\,W_{\alpha,k-\frac{1}{2}}\,.$$

Then

$$\left(1 + \frac{z}{2-z}\right)W_{\alpha-1,k} = (1-z)^{-\frac{1}{2}}\Big[W_{\alpha-1,k-\frac{1}{2}} + \frac{(\alpha-2k)z}{2(2\alpha-1)(z-2)}\,W_{\alpha,k-\frac{1}{2}}\Big]\,.$$

Solving for $W_{\alpha-1,k}$, we have the recurrence relation

$$W_{\alpha-1,k} = \frac{(2-z)}{2}\,(1-z)^{-\frac{1}{2}}\Big[W_{\alpha-1,k-\frac{1}{2}} + \frac{(\alpha-2k)z}{2(2\alpha-1)(z-2)}\,W_{\alpha,k-\frac{1}{2}}\Big]\,,$$

or

$$(2\text{-}42)\quad W_{\alpha-1,k} = \frac{1}{4(1-2\alpha)}(1-z)^{\frac{1}{2}}\Big[2(2\alpha-1)(z-2)\,W_{\alpha-1,k-\frac{1}{2}} + (\alpha-2k)z\,W_{\alpha,k-\frac{1}{2}}\Big].$$

Another recurrence relation is found by first replacing α by $\alpha + 1$ in (2-30), giving us

$$W_{\alpha+1,k} = z^{\alpha+1}(1-z)^{k+\frac{1}{2}}\,F(\alpha+1,\ \alpha+2k+1;\ 2\alpha+2;\ z).$$

Then apply (2-33) to get

$$W_{\alpha+1,k} = z^{\alpha+1}(1-z)^{k+\frac{1}{2}}\,\frac{(2\alpha+1)}{\alpha z}\Big[F(\alpha,\ \alpha+2k+1;\ 2\alpha+1;\ z) - F(\alpha,\ \alpha+2k;\ 2\alpha+1;\ z)\Big].$$

When (2-31) is applied to each term, we may write

$$W_{\alpha+1,k} = \frac{(2\alpha+1)}{\alpha}\,z^{\alpha}(1-z)^{k+\frac{1}{2}}\left\{\frac{1}{-\alpha}\Big[\alpha F(\alpha+1,\ \alpha+2k+1;\ 2\alpha+1;\ z) - 2\alpha F(\alpha,\ \alpha+2k+1;\ 2\alpha;\ z)\Big] + \frac{1}{\alpha}\Big[\alpha F(\alpha+1,\ \alpha+2k;\ 2\alpha+1;\ z) - 2\alpha F(\alpha,\ \alpha+2k;\ 2\alpha;\ z)\Big]\right\}.$$

$$(2\text{-}43)\quad W_{\alpha+1,k} = \frac{(2\alpha+1)}{\alpha}\,z^{\alpha}(1-z)^{k+\frac{1}{2}}\Big[-F(\alpha+1,\ \alpha+2k+1;\ 2\alpha+1;\ z) + 2F(\alpha,\ \alpha+2k+1;\ 2\alpha;\ z) + F(\alpha+1,\ \alpha+2k;\ 2\alpha+1;\ z) - 2F(\alpha,\ \alpha+2k;\ 2\alpha;\ z)\Big].$$

From (2-35), letting $n = 1$, we have

$$F(+1) = 2z^{-\alpha-1}(1-z)^{-k-1}\left[W_{\alpha,k+\frac{1}{2}} - (1-z)^{\frac{1}{2}}W_{\alpha,k}\right],$$

which can be used to rewrite the first and third terms of (2-43). Hence,

$$W_{\alpha+1,k} = \frac{(2\alpha+1)}{\alpha} z^{\alpha}(1-z)^{k+\frac{1}{2}}\left\{-2z^{-\alpha-1}(1-z)^{-k-1}\left[W_{\alpha,k+\frac{1}{2}}\right.\right.$$

$$\left. - (1-z)^{\frac{1}{2}} W_{\alpha,k}\right] + 2z^{-\alpha}(1-z)^{-k-1} W_{\alpha,k+\frac{1}{2}}$$

$$+ 2z^{-\alpha-1}(1-z)^{-k-\frac{1}{2}}\left[W_{\alpha,k} - (1-z)^{\frac{1}{2}} W_{\alpha,k-\frac{1}{2}}\right]$$

$$\left. - 2z^{-\alpha}(1-z)^{-k-\frac{1}{2}} W_{\alpha,k}\right\}$$

$$= \frac{2(2\alpha+1)}{\alpha}\left\{\left[-z^{-1}(1-z)^{-\frac{1}{2}} + (1-z)^{-\frac{1}{2}}\right] W_{\alpha,k+\frac{1}{2}}\right.$$

$$\left. + \left[2z^{-1} - 1\right] W_{\alpha,k} - z^{-1}(1-z)^{\frac{1}{2}} W_{\alpha,k-\frac{1}{2}}\right\}$$

$$= \frac{2(2\alpha+1)}{\alpha z}\left[-(1-z)^{\frac{1}{2}} W_{\alpha,k+\frac{1}{2}} + (2-z) W_{\alpha,k} - (1-z)^{\frac{1}{2}} W_{\alpha,k-\frac{1}{2}}\right].$$

Thus, we have a second recurrence relation

$$\alpha z W_{\alpha+1,k} = 2(2\alpha+1)\left[-(1-z)^{\frac{1}{2}} W_{\alpha,k+\frac{1}{2}} + (2-z) W_{\alpha,k}\right. \tag{2-44}$$

$$\left. - (1-z)^{\frac{1}{2}} W_{\alpha,k-\frac{1}{2}}\right].$$

To obtain the nth derivative of our solution (2-30)

$$W_{\alpha,k} = z^{\alpha}(1-z)^{k+\frac{1}{2}} F(\alpha, \alpha+2k; 2\alpha; z),$$

we shall apply Leibniz's rule, where $u = z^{\alpha}(1-z)^{k+\frac{1}{2}}$ and $v = F(\alpha, \alpha+2k; 2\alpha; z)$. Thus, using (2-34) thru (2-36) we may write

$$D^n W_{\alpha,k} = \sum_{\ell=0}^{n} \binom{n}{\ell} D^{\ell}\left[z^{\alpha}(1-z)^{k+\frac{1}{2}}\right] D^{n-\ell}\left[F(\alpha, \alpha+2k; 2\alpha; z)\right]$$

$$= \sum_{\ell=0}^{n}\left[\binom{n}{\ell} z^{\alpha}(1-z)^{k-\ell+\frac{1}{2}} \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} (-\alpha)_j (-k-\tfrac{1}{2})_{\ell-j} z^{-j}(1-z)^j \right.$$

$$\left. \cdot \frac{(\alpha)_{n-\ell}(\alpha+2k)_{n-\ell}}{(2\alpha)_{n-\ell}} \; F(+n-\ell)\right]$$

$$= \sum_{\ell=0}^{n}\left[\binom{n}{\ell} z^{\alpha}(1-z)^{k-\ell+\frac{1}{2}} \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} (-\alpha)_j (-k-\tfrac{1}{2})_{\ell-j} z^{-j}(1-z)^j \right.$$

$$\left. \cdot (\alpha+2k)_{n-\ell} z^{-\alpha-n+\ell}(1-z)^{-k-\frac{n-\ell+1}{2}} \sum_{j=0}^{n-\ell} (-1)^j \binom{n-\ell}{j} (1-z)^{\frac{j}{2}} W_{\alpha,k+\frac{n-\ell-j}{2}}\right].$$

Therefore,

$$(2\text{-}45)\quad D^n W_{\alpha,k} = z^{-n}(1-z)^{-\frac{n}{2}} \sum_{\ell=0}^{n}\left[\binom{n}{\ell}(1-z)^{-\frac{\ell}{2}} z^{\ell}(\alpha+2k)_{n-\ell}\right.$$

$$\cdot \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} (-\alpha)_j (-k-\tfrac{1}{2})_{\ell-j} z^{-j}(1-z)^j$$

$$\left. \cdot \sum_{j=0}^{n-\ell} (-1)^j \binom{n-\ell}{j} (1-z)^{\frac{j}{2}} W_{\alpha,k+\frac{n-\ell-j}{2}}\right].$$

Now that we have the nth derivative of $W_{\alpha,k}(z)$ we can use Taylor's Theorem [8, p. 93] to write formulas for $W_{\alpha,k}(x+y)$ and $W_{\alpha,k}(xy)$. Taylor's expansion can be expressed in the form

$$f(y+a) = f(a) + f'(a)y + f''(a)\frac{y^2}{2!} + \ldots + f^{(n)}(a)\frac{y^n}{n!} + \ldots .$$

Now if a be replaced by x, we have

$$f(y+x) = f(x) + f'(x)y + f''(x)\frac{y^2}{2!} + \ldots + f^{(n)}(x)\frac{y^n}{n!} + \ldots .$$

$$f(y+x) = \sum_{n=0}^{\infty} f^{(n)}(x)\frac{y^n}{n!} . \tag{2-46}$$

Using (2-46), replace y by (y-1)x to obtain

$$f(xy) = \sum_{n=0}^{\infty} f^{(n)}(x)\frac{(y-1)^n x^n}{n!} . \tag{2-47}$$

Since $W_{\alpha,k}(z)$ is a convergent, analytic function for $0 < |z| < 1$, it can be used in place of f(x) above and we have an addition theorem and a multiplication theorem:

$$W_{\alpha,k}(x+y) = \sum_{n=0}^{\infty} W_{\alpha,k}^{(n)}(x)\frac{y^n}{n!} \tag{2-48}$$

$$= \sum_{n=0}^{\infty} \left\{ \frac{y^n}{n!} x^{-n}(1-x)^{-\frac{n}{2}} \sum_{\ell=0}^{n} \left[\binom{n}{\ell} (1-x)^{-\frac{\ell}{2}} x^{\ell} (\alpha+2k)_{n-\ell} \right.\right.$$

$$\left.\left. \cdot \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} (-\alpha)_j \left(-k-\tfrac{1}{2}\right)_{\ell-j} x^{-j} (1-x)^j \right.\right.$$

$$\cdot \sum_{j=0}^{n-\ell} (-1)^j \binom{n-\ell}{j} (1-x)^{\frac{j}{2}} W_{\alpha,k+\frac{n-\ell-j}{2}} \Bigg] \Bigg\} ,$$

and

$$(2\text{-}49) \quad W_{\alpha,k}(xy) = \sum_{n=0}^{\infty} W_{\alpha,k}^{(n)}(x) \frac{(y-1)^n x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \Bigg\{ \frac{(y-1)^n}{n!} (1-x)^{-\frac{n}{2}} \sum_{\ell=0}^{n} \Bigg[\binom{n}{\ell} (1-x)^{-\frac{\ell}{2}} x^{\ell} (\alpha+2k)_{n-\ell}$$

$$\cdot \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} (-\alpha)_j (-k-\tfrac{1}{2})_{\ell-j} x^{-j} (1-x)^j$$

$$\cdot \sum_{j=0}^{n-\ell} (-1)^j \binom{n-\ell}{j} (1-x)^{\frac{j}{2}} W_{\alpha,k+\frac{n-\ell-j}{2}} \Bigg] \Bigg\} ,$$

where $|y| < 1$.

CHAPTER III

Related Differential Equations and Classification

3.1 Related Equations of Mathematical Physics Derived from an Equation having Four Singularities. The most general linear differential equation of the second order which has every point except a_1, a_2, a_3 and ∞ as an ordinary point with exponents α_r, β_r at a_r $(r=1,2,3)$ and exponents μ_1, μ_2 at ∞, is

$$(3\text{-}1)\quad \frac{d^2u}{dz^2} + \sum_{r=1}^{3} \frac{1-\alpha_r-\beta_r}{z-a_r} \cdot \frac{du}{dz} + \left[\sum_{r=1}^{3} \frac{\alpha_r\beta_r}{(z-a_r)^2} + \frac{Az+B}{\prod_{r=1}^{3}(z-a_r)} \right] u = 0,$$

where A is such that μ_1 and μ_2 are the roots of

$$(3\text{-}2)\qquad \mu^2 + \mu\left[\sum_{r=1}^{3} (\alpha_r+\beta_r) - 2 \right] + \sum_{r=1}^{3} \alpha_r\beta_r + A = 0.$$

To verify this, we can begin with equation (2-1) and let a_4 become infinite, thus obtaining

$$u'' + \left[\sum_{r=1}^{3} \frac{1-\alpha_r-\beta_r}{z-a_r} + H(z) \right] u' + \left\{ \sum_{r=1}^{3} \left[\frac{\alpha_r\beta_r}{(z-a_r)^2} + \frac{m_r}{z-a_r} \right] + J(z) \right\} u = 0,$$

where $H(z)$ and $J(z)$ are polynomials in z. We must now depart from the previous work of chapter II, for we desire $z = \infty$ to be a regular singular point in this case. Referring to section 1.5, we see that $z = \infty$ $\left(z = \frac{1}{z_1}\right)$ will be a regular

singular point when $2 - \frac{1}{z_1} p\left(\frac{1}{z_1}\right)$ and $\frac{1}{z_1^2} q\left(\frac{1}{z_1}\right)$ are analytic at $z_1 = 0$. For the first one,

$$2 - \frac{1}{z_1} p\left(\frac{1}{z_1}\right) = 2 - \frac{1}{z_1}\left[\frac{1-\alpha_1-\beta_1}{\frac{1}{z_1} - a_1} + \frac{1-\alpha_2-\beta_2}{\frac{1}{z_1} - a_2} + \frac{1-\alpha_3-\beta_3}{\frac{1}{z_1} - a_3} + H\left(\frac{1}{z_1}\right)\right]$$

$$= 2 - \left[\frac{1-\alpha_1-\beta_1}{1-a_1 z_1} + \frac{1-\alpha_2-\beta_2}{1-a_2 z_1} + \frac{1-\alpha_3-\beta_3}{1-a_3 z_1} + \frac{1}{z_1} H\left(\frac{1}{z_1}\right)\right],$$

which is analytic at $z_1 = 0$ if $H\left(\frac{1}{z_1}\right) \equiv 0$. In the second,

$$\frac{1}{z_1^2} q\left(\frac{1}{z_1}\right) = \frac{1}{z_1^2}\left[\frac{\alpha_1\beta_1}{\left(\frac{1}{z_1} - a_1\right)^2} + \frac{\alpha_2\beta_2}{\left(\frac{1}{z_1} - a_2\right)^2} + \frac{\alpha_3\beta_3}{\left(\frac{1}{z_1} - a_3\right)^2} + \frac{m_1}{\frac{1}{z_1} - a_1}\right.$$

$$\left. + \frac{m_2}{\frac{1}{z_1} - a_2} + \frac{m_3}{\frac{1}{z_1} - a_3} + J\left(\frac{1}{z_1}\right)\right]$$

$$= \frac{\alpha_1\beta_1}{1-2a_1 z_1 + a_1^2 z_1^2} + \frac{\alpha_2\beta_2}{1-2a_2 z_1 + a_2^2 z_1^2} + \frac{\alpha_3\beta_3}{1-2a_3 z_1 + a_3^2 z_1^2}$$

$$+ \frac{1}{z_1}\left[\frac{m_1}{1-a_1 z_1} + \frac{m_2}{1-a_2 z_1} + \frac{m_3}{1-a_3 z_1}\right] + \frac{1}{z_1^2} J\left(\frac{1}{z_1}\right),$$

which is analytic at $z_1 = 0$, only if $m_1 + m_2 + m_3 = 0$ and $J\left(\frac{1}{z_1}\right) = 0$.

Now

$$\frac{m_1}{z-a_1} + \frac{m_2}{z-a_2} + \frac{m_3}{z-a_3}$$

$$= \left[(m_1+m_2+m_3)z^2 + Az + B\right]\frac{1}{(z-a_1)(z-a_2)(z-a_3)},$$

where A and B are functions of m_r and a_r ($r = 1,2,3$). But since $m_1+m_2+m_3 = 0$, we have equation (3-1). In order to show that A must satisfy the condition (3-2), we can make the substitution $z = \frac{1}{z_1}$ in (3-1), then the indicial equation for $z_1 = 0$ will be this condition. An alternative method of finding the indicial equation at $z = \infty$ (or $z_1 = 0$), and the procedure we shall use here, would be to assume the solution

$$u = b_o z^{-\lambda} + b_1 z^{-\lambda-1} + \ldots$$

and equate the coefficient of the largest power of z (i.e. $z^{-\lambda-2}$) to zero to obtain the indicial equation.

Noting the expansion of the following about $z = \infty$(or $\frac{1}{z} = 0$), we have

$$\frac{1}{z-a_r} = \frac{1}{z}\left(\frac{1}{1-\frac{a_r}{z}}\right) = \frac{1}{z}\left(1+\frac{a_r}{z}+\frac{a_r^2}{z^2}+\ldots\right),$$

$$\frac{1}{(z-a_r)^2} = \frac{1}{z^2}\left(1+\frac{a_r}{z}+\frac{a_r^2}{z^2}+\ldots\right)^2 = \frac{1}{z^2}\left(1+\frac{2a_r}{z}+\ldots\right),$$

where $r = 1,2,3$, and

$$\frac{1}{(z-a_1)(z-a_2)(z-a_3)} = \frac{1}{z^3}\left(\frac{1}{1-\frac{a_1}{z}}\right)\left(\frac{1}{1-\frac{a_2}{z}}\right)\left(\frac{1}{1-\frac{a_3}{z}}\right)$$

$$= \frac{1}{z^3}\left(1+\frac{a_1+a_2+a_3}{z}+\ldots\right).$$

Then

$$\sum_{r=1}^{3} \frac{\alpha_r\beta_r}{z^2}\left(1+\frac{2a_r}{z}+\ldots\right) + \frac{Az+B}{z^3}\left(1+\frac{a_1+a_2+a_3}{z}+\ldots\right) \qquad \left| \; u = b_o z^{-\lambda}+b_1 z^{-\lambda-1}+\ldots, \quad b_o \neq 0. \right.$$

$$\sum_{r=1}^{3} \frac{1-\alpha_r-\beta_r}{z}\left(1+\frac{a_r}{z}+\frac{a_r^2}{z^2}+\ldots\right) \qquad \left| \; u' = -b_o\lambda z^{-\lambda-1}+b_1(-\lambda-1)z^{-\lambda-2}+\ldots. \right.$$

$$1 \qquad \left| \; u'' = b_o\lambda(\lambda+1)z^{-\lambda-2} + b_1(\lambda+1)(\lambda+2)z^{-\lambda-3}+\ldots. \right.$$

$$z^{-\lambda-2}: \quad \lambda(\lambda+1) - \lambda\sum_{r=1}^{3}(1-\alpha_r-\beta_r) + \sum_{r=1}^{3}\alpha_r\beta_r + A = 0.$$

$$\lambda^2 + \left[1-\sum_{r=1}^{3}(1-\alpha_r-\beta_r)\right]\lambda + \sum_{r=1}^{3}\alpha_r\beta_r + A = 0.$$

$$\lambda^2 + \left[\sum_{r=1}^{3}(\alpha_r+\beta_r)-2\right]\lambda + \sum_{r=1}^{3}\alpha_r\beta_r + A = 0,$$

where the exponents at $z = \infty$ are $\lambda = \mu_1$ and μ_2.

We now proceed to show how many of the related differential equations of mathematical physics can be obtained from a differential equation having four singularities, by assigning values to the exponents α_r, β_r, the singularities a_r $(r = 1,2,3)$, and the constant B.

(a) Lame's Equation [8, p. 205]. Beginning with (3-1) take $\alpha_r = 1/2$ and $\beta_r = 0$ $(r = 1,2,3)$, $\mu_2 = \mu_1 + n + \frac{1}{2}$, and

$B = -\frac{1}{4}h$, where h and n are constants. We can obtain A by using (3-2) and the fact that the sum of the roots equals $2 - \sum_{r=1}^{3}(\alpha_r+\beta_r) = \frac{1}{2}$ and the product of the roots is $\sum_{r=1}^{3}\alpha_r\beta_r + A = A$, where the roots are μ_1 and $\mu_1 + n + \frac{1}{2}$. Thus,

$$2\mu_1 + n + \tfrac{1}{2} = \tfrac{1}{2}, \qquad \mu_1 = -\tfrac{1}{2}n,$$

and

$$A = \mu_1(\mu_1+n+\tfrac{1}{2}) = -\tfrac{1}{2}\,n(\tfrac{1}{2}n+\tfrac{1}{2}) = -\tfrac{1}{4}\,n(n+1).$$

Substituting these values into (3-1), the result is Lame's Equation:

$$\frac{d^2u}{dz^2} + \sum_{r=1}^{3}\frac{\frac{1}{2}}{z-a_r}\,\frac{du}{dz} - \frac{n(n+1)z + h}{4\prod_{r=1}^{3}(z-a_r)}\,u = 0.$$

(b) Legendre's Equation. Let $a_1 = a_2 = 0$, $a_3 = 1$, $\alpha_3 = \beta_3 = 0$, $\mu_1 = 0$, $\mu_2 = \frac{1}{2}$, $\alpha_1+\beta_1+\alpha_2+\beta_2 = 3/2$, $B = \frac{1}{4}n(n+1) + \alpha_1\beta_1 + \alpha_2\beta_2$ in (3-1). Since the product $\mu_1\mu_2 = 0 = \sum_{r=1}^{3}\alpha_r\beta_r + A$ from (3-2), we have $A = -(\alpha_1\beta_1 + \alpha_2\beta_2)$. Hence,

$$\frac{d^2u}{dz^2} + \left[\frac{\frac{1}{2}}{z} + \frac{1}{z-1}\right]\frac{du}{dz} + \left[\frac{\alpha_1\beta_1 + \alpha_2\beta_2}{z^2} + \frac{-(\alpha_1\beta_1 + \alpha_2\beta_2)z + \frac{1}{4}n(n+1) + \alpha_1\beta_1 + \alpha_2\beta_2}{z^2(z-1)}\right]u = 0.$$

$$(3\text{-}3) \qquad \frac{d^2u}{dz^2} + \left[\frac{\frac{1}{2}}{z} + \frac{1}{z-1}\right]\frac{du}{dz} + \frac{n(n+1)}{4z^2(z-1)}\,u = 0.$$

Now let $z = \zeta^{-2}$.

$$dz = -2\ \zeta^{-3} d\zeta, \qquad \frac{d\zeta}{dz} = -\tfrac{1}{2}\ \zeta^{3}\ .$$

Then

$$\frac{du}{dz} = \frac{du}{d\zeta}\frac{d\zeta}{dz} = -\tfrac{1}{2}\ \zeta^{3}\ \frac{du}{d\zeta}\ .$$

$$\frac{d^2u}{dz^2} = \left[-\frac{3}{2}\ \zeta^{2}\ \frac{du}{d\zeta} - \tfrac{1}{2}\ \zeta^{3}\ \frac{d^2u}{d\zeta^2}\right](-\tfrac{1}{2}\ \zeta^{3}) = \frac{3}{4}\ \zeta^{5}\ \frac{du}{d\zeta} + \frac{1}{4}\ \zeta^{6}\ \frac{d^2u}{d\zeta^2}\ .$$

Substituting in (3-3):

$$\frac{1}{4}\ \zeta^{6}\ \frac{d^2u}{d\zeta^2} + \left[\frac{3}{4}\ \zeta^{5} - \frac{1}{4}\ \zeta^{5} - \frac{\frac{1}{2}\ \zeta^{5}}{1-\zeta^{2}}\right]\frac{du}{d\zeta} + \frac{n(n+1)\zeta^{6}}{4(1-\zeta^{2})}\ u = 0.$$

$$\frac{1}{4}\ \zeta^{6}\ \frac{d^2u}{d\zeta^2} - \frac{\frac{1}{2}\zeta^{7}}{1-\zeta^{2}}\ \frac{du}{d\zeta} + \frac{n(n+1)\zeta^{6}}{4(1-\zeta^{2})}\ u = 0.$$

Multiplying by $\dfrac{4(1-\zeta^{2})}{\zeta^{6}}$, we have the General Legendre equation:

$$(1-\zeta^{2})\ \frac{d^2u}{d\zeta^2} - 2\zeta\ \frac{du}{d\zeta} + n(n+1)\ u = 0.$$

(c) Jacobi's Equation. If we let A be an arbitrary finite constant, $B = a_3 n(\alpha+\beta+n+1)$, $\alpha_1 = -\alpha$, $\alpha_2 = -\beta$, $\beta_1 = \beta_2 = 0$, $a_1 = 1$, $a_2 = -1$ in equation (3-1), then

$$\frac{d^2u}{dz^2} + \left[\frac{1+\alpha}{z-1} + \frac{1+\beta}{z+1} + \frac{1-\alpha_3-\beta_3}{z-a_3}\right]\frac{du}{dz}$$

$$+\left[\frac{\alpha_3\beta_3}{(z-a_3)^2} + \frac{Az+a_3n(\alpha+\beta+n+1)}{(z-1)(z+1)(z-a_3)}\right]u = 0.$$

Now let a_3 become infinite and we have

$$\frac{d^2u}{dz^2} + \left[\frac{1+\alpha}{z-1} + \frac{1+\beta}{z+1}\right]\frac{du}{dz} - \frac{n(\alpha+\beta+n+1)}{(z-1)(z+1)}\,u = 0.$$

This can be written in the form of Jacobi's equation:

$$\frac{d^2u}{dz^2} + \frac{\beta-\alpha-(\alpha+\beta+2)z}{1-z^2}\frac{du}{dz} + \frac{n(\alpha+\beta+n+1)}{1-z^2}\,u = 0.$$

(d) Gegenbauer's Equation [8, p. 329]. We shall first derive Gegenbauer's equation which is

$$(1-z^2)u'' - (1+2\nu)zu' + n(n+2\nu)u = 0,$$

and then show how it can be obtained from equation (3-1).

Let ϕ be defined:

$$\phi = (1-2hz+h^2)^{-\nu} = \sum_{n=0}^{\infty} C_n(z)\,h^n. \tag{3-4}$$

Next, find $\frac{\partial\phi}{\partial h}$ and $\frac{\partial\phi}{\partial z}$.

$$\frac{\partial\phi}{\partial h} = -\nu(1-2hz+h^2)^{-\nu-1}(-2z+2h)$$

$$= 2\nu(z-h)(1-2hz+h^2)^{-\nu-1},$$

hence

$$\frac{\partial\phi}{\partial h} = 2\nu(z-h)\phi(1-2hz+h^2)^{-1}. \tag{3-5}$$

$$\frac{\partial\phi}{\partial z} = 2h\nu(1-2hz+h^2)^{-\nu-1},$$

$$\frac{\partial\phi}{\partial z} = 2h\nu\,\phi(1-2hz+h^2)^{-1}. \tag{3-6}$$

Then from (3-5) and (3-6) we have

(3-7) $$h \frac{\partial \phi}{\partial h} = (z-h) \frac{\partial \phi}{dz} .$$

Using the form

$$\phi = \sum_{n=0}^{\infty} C_n h^n,$$

we may also write

(3-8) $$\frac{\partial \phi}{\partial h} = \sum_{n=1}^{\infty} nC_n h^{n-1}$$

and

(3-9) $$\frac{\partial \phi}{\partial z} = \sum_{n=0}^{\infty} C_n' h^n .$$

Rewriting (3-7) with the use of (3-8) and (3-9) gives us

$$\sum_{n=1}^{\infty} nC_n h^n = (z-h) \sum_{n=0}^{\infty} C_n' h^n .$$

Equating the coefficients of h^{n-1} and h^n we obtain:

h^{n-1}: $$(n-1)C_{n-1} = zC_{n-1}' - C_{n-2}' , \qquad n=2,3,\ldots,$$

(3-10) $$C_{n-2}' = zC_{n-1}' - (n-1)C_{n-1} .$$

In (3-10), replace n by n+1, thus obtaining

(3-11) $$C_{n-1}' = zC_n' - nC_n , \qquad n=1,2,\ldots.$$

Using (3-8) in (3-5) and letting $\phi = \sum_{n=0}^{\infty} C_n h^n$, we now have

$$\sum_{n=1}^{\infty} nC_n h^{n-1} = 2\nu(z-h)(1-2hz+h^2)^{-1} \sum_{n=0}^{\infty} C_n h^n ,$$

$$(1-2hz+h^2) \sum_{n=1}^{\infty} nC_n h^{n-1} = 2\nu(z-h) \sum_{n=0}^{\infty} C_n h^n .$$

Equating coefficients of h^{n-1},

$$nC_n - 2(n-1)zC_{n-1} + (n-2)C_{n-2} = 2\nu zC_{n-1} - 2\nu C_{n-2},$$

and

$$nC_n - 2(n-1+\nu)zC_{n-1} + (n-2+2\nu)C_{n-2} = 0. \tag{3-12}$$

Differentiating (3-12) with respect to z gives us

$$nC_n' - 2(n-1+\nu)C_{n-1} - 2(n-1+\nu)zC_{n-1}' + (n-2+2\nu)C_{n-2}' = 0.$$

Using (3-10) and simplifying,

$$nC_n' - 2(n-1+\nu)C_{n-1} - 2(n-1+\nu)zC_{n-1}' + (n-2+2\nu)\left[zC_{n-1}' - (n-1)C_{n-1}\right] = 0,$$

$$nC_n' - \left[2(n-1+\nu) + (n-1)(n-2+2\nu)\right] C_{n-1} + \left[-2(n-1+\nu) + (n-2+2\nu)\right] zC_{n-1}' = 0,$$

$$nC_n' - n(n+2\nu-1)C_{n-1} - nzC_{n-1}' = 0,$$

$$C_n' - (n+2\nu-1)C_{n-1} - zC_{n-1}' = 0.$$

Now use (3-11) and simplify the result:

$$C_n' - (n+2\nu-1)C_{n-1} - z^2C_n' + nzC_n = 0,$$

$$(3\text{-}13) \qquad (1-z^2)C_n' - (n+2\nu-1)C_{n-1} + nzC_n = 0.$$

Differentiating (3-13) with respect to z, we have

$$-2zC_n' + (1-z^2)C_n'' - (n+2\nu-1)C_{n-1}' + nC_n + nzC_n' = 0,$$

$$(1-z^2)C_n'' + (n-2)zC_n' - (n+2\nu-1)C_{n-1}' + nC_n = 0.$$

With the use of (3-11), we may write

$$(1-z^2)C_n'' + (n-2)zC_n' - (n+2\nu-1)(zC_n'-nC_n) + nC_n = 0,$$

$$(1-z^2)C_n'' - (1+2\nu)zC_n' + n(n+2\nu)C_n = 0.$$

If C_n is replaced by u, we have the desired form of Gegenbauer's equation,

$$(1-z^2)u'' - (1+2\nu)zu' + n(n+2\nu)u = 0.$$

If we then let $\nu = \frac{1}{2}$, we have Legendre's equation,

$$(1-z^2)u'' - 2zu' + n(n+1)u = 0.$$

Now to obtain Gegenbauer's equation, we use equation (3-1), letting $a_1 = 1, a_2 = -1, \alpha_1 = \alpha_2 = \frac{1}{2} - \nu$, $\beta_1 = \beta_2 = 0$, A is an arbitrary finite constant, $B = a_3 n(n+2\nu)$. Thus,

$$\frac{d^2u}{dz^2} + \left[\frac{1-\frac{1}{2}+\nu}{z-1} + \frac{1-\frac{1}{2}+\nu}{z+1} + \frac{1-\alpha_3-\beta_3}{z-a_3}\right]\frac{du}{dz}$$

$$+ \left[\frac{\alpha_3\beta_3}{(z-a_3)^2} + \frac{Az+a_3n(n+2\nu)}{(z-1)(z+1)(z-a_3)}\right]u = 0.$$

Now let a_3 become infinite and we have the desired result:

$$\frac{d^2u}{dz^2} + \frac{(1+2\nu)z}{z^2-1}\frac{du}{dz} - \frac{n(n+2\nu)}{z^2-1}u = 0,$$

$$(1-z^2)\frac{d^2u}{dz^2} - (1+2\nu)z\frac{du}{dz} + n(n+2\nu)u = 0.$$

Before continuing further let us obtain the equation to be used in subsequent cases. Using the scheme

$$u = \left\{ \begin{matrix} a_1 & a_2 & a_3 & \infty & \\ a_1\ell+f & \alpha_2 & \alpha_3 & -a_1\ell - \frac{1}{a_1} + t & z \\ \frac{1}{a_1} & \beta_2 & \beta_3 & \beta_4 & \end{matrix} \right\}$$

we can write equation (2-2):

$$u'' + \left[\frac{1-a_1\ell-f-\frac{1}{a_1}}{z-a_1} + \frac{1-\alpha_2-\beta_2}{z-a_2} + \frac{1-\alpha_3-\beta_3}{z-a_3} \right] u'$$

$$+ \left[\frac{(\ell+\frac{f}{a_1})(a_1-a_2)(a_1-a_3)}{(z-a_1)^2(z-a_2)(z-a_3)} + \frac{\alpha_2\beta_2(a_2-a_3)}{(z-a_2)^2(z-a_3)} + \frac{\alpha_3\beta_3(a_3-a_1)}{(z-a_3)^2(z-a_1)} \right.$$

$$\left. + \frac{\beta_4(-a_1\ell-\frac{1}{a_1}+t)}{(z-a_1)(z-a_2)} \right] u = 0,$$

where $f + \alpha_2 + \beta_2 + \alpha_3 + \beta_3 + t + \beta_4 = 2$; f, ℓ and t are arbitrary constants. Now let a_1 become infinite, then

$$(3\text{-}14) \quad u'' + \left[\ell + \frac{1-\alpha_2-\beta_2}{z-a_2} + \frac{1-\alpha_3-\beta_3}{z-a_3} \right] u' + \left[\frac{\ell}{(z-a_2)(z-a_3)} \right.$$

$$\left. + \frac{\alpha_2\beta_2(a_2-a_3)}{(z-a_2)^2(z-a_3)} + \frac{\alpha_3\beta_3}{(z-a_3)^2} + \frac{\ell\beta_4}{(z-a_2)} \right] u = 0,$$

and there are no restrictions placed on the arbitrary constants.

(e) Laguerre's Equation [3, p. 186]. The equation is

$$zu'' + (1+\alpha-z)u' + nu = 0.$$

If we assign the following values to the arbitrary constants in equation (3-14), we have the desired differential equation: $a_2 = a_3 = 0$, $\alpha_2 = -\alpha$, $\beta_2 = 0$, $\ell = -1$, $\alpha_3 + \beta_3 = 1$, $\alpha_3\beta_3 = 1$, $\beta_4 = -n$.

$$u'' + \left[-1 + \frac{1+\alpha}{z}\right] u' + \left[\frac{-1}{z^2} + \frac{1}{z^2} + \frac{n}{z}\right] u = 0,$$

thus

$$zu'' + (1+\alpha-z)u' + nu = 0.$$

(f) <u>The Equation having the Incomplete Gamma Functions as Solutions</u> [7, p. 96]. The equation is

$$zu'' + (1+a+z)u' + au = 0,$$

with solutions

$$\gamma(a,z) = \frac{z^a}{a}\, {}_1F_1(a;\ 1+a;\ -z).$$

Using equation (3-14), let $a_2 = a_3 = 0$, $\alpha_2 = 1-a$, $\beta_2 = 0$, $\alpha_3 = -\beta_3 = 1$, $\beta_4 = a$, $\ell = 1$, then

$$u'' + \left[1 + \frac{a}{z} + \frac{1}{z}\right] u' + \left[\frac{1}{z^2} - \frac{1}{z^2} + \frac{a}{z}\right] u = 0,$$

$$zu'' + (1+a+z)u' + au = 0.$$

(g) <u>Gauss's Equation</u> [7, p. 2]. Consider equation (3-1) with the substitutions: $a_1 = a_2 = 0$, $a_3 = 1$, $\alpha_1 = A+1-ab-c$, $\alpha_2 = B = ab-A$, $\alpha_3 = -a-b+c$, $\beta_1 = \beta_3 = 0$, $\beta_2 = 1$.

$$u'' + \left[\frac{ab+c-A}{z} + \frac{A-ab}{z} + \frac{1+a+b-c}{z-1}\right] u' + \left[\frac{ab-A}{z^2} + \frac{Az+ab-A}{z^2(z-1)}\right] u = 0.$$

$$u'' + \left[\frac{c(z-1)+(1+a+b-c)z}{z(z-1)}\right] u' + \left[\frac{(ab-A)(z-1)+Az+ab-A}{z^2(z-1)}\right] u = 0.$$

$$u'' + \left[\frac{c-(1+a+b)z}{z(1-z)}\right] u' - \frac{ab}{z(1-z)} u = 0.$$

Therefore,

$$z(1-z)u'' + \left[c-(1+a+b)z\right] u' - abu = 0,$$

which is Gauss's differential equation, often called the hypergeometric equation, and has the hypergeometric function ${}_2F_1(a, b; c; z)$ as a solution [8, p. 283].

(h) Kummer's Equation [7, p. 2]. Use equation (3-14) with $a_2 = a_3 = 0$, $\alpha_2 = -b$, $\beta_2 = 0$, $\alpha_3 = \beta_3 = 1$, $\beta_4 = a$, $\ell = -1$, then

$$u'' + \left[-1+\frac{1+b}{z} - \frac{1}{z}\right] u' + \left[\frac{-1}{z^2} + \frac{1}{z^2} - \frac{a}{z}\right] u = 0.$$

Hence,

$$zu'' + (b-z)u' - au = 0,$$

which is the confluent hypergeometric equation known as Kummer's equation, and any solution of this equation is a confluent hypergeometric function ${}_1F_1(a; b; z)$.

(i) Whittaker's Equation [8, p. 337]. The equation studied by Whittaker is

$$W'' + \left[-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4}-m^2}{z^2}\right] W = 0,$$

and one of its solutions is Whittaker's function

$$M_{k,m}(z) = z^{\frac{1}{2}+m} e^{-\frac{1}{2}z} \; {}_1F_1(\tfrac{1}{2}+m-k; 1+2m; z).$$

Any general solution of Whittaker's equation is called a Whittaker function and can be expressed in terms of a confluent hypergoemetric function ${}_1F_1(a;b;z)$.

We begin with equation (3-14), letting $a_2 = a_3 = 0$, $\alpha_2 = \frac{9}{4}$, $\beta_2 = -m^2$, $\alpha_3 = \frac{3}{4} + m^2$, $\beta_3 = -1$, $\beta_4 = k$, $\ell = 1$, so that

$$u'' + \left[1 + \frac{1 - \frac{9}{4} + m^2}{z} + \frac{1 - \frac{3}{4} - m^2 + 1}{z}\right] u' + \left[\frac{1}{z^2} - \frac{\frac{3}{4} + m^2}{z^2} + \frac{k}{z}\right] u = 0.$$

Thence

$$u'' + u' + \left(\frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right) u = 0. \tag{3-15}$$

To normalize (3-15) use (2-19) to find

$$v = z^{-\frac{1}{2}}.$$

Then

$$v' = -\tfrac{1}{2} z^{-\frac{3}{2}}.$$

$$v'' = \frac{3}{4} z^{-\frac{5}{2}}.$$

Equation (2-20) now becomes

$$z^{\frac{3}{2}} w'' + \left(\frac{3}{4} z^{-\frac{1}{2}} - \frac{1}{2} z^{-\frac{1}{2}} + kz^{\frac{1}{2}} + \frac{1}{4} z^{-\frac{1}{2}} - m^2 z^{-\frac{1}{2}}\right) w = 0.$$

Upon simplifying and multiplying by $z^{\frac{1}{2}}$, the equation can be written

$$z^2 w'' + (-m^2 + kz) w = 0,$$

which is Whittaker's differential equation.

3.2 Classification of Differential Equations having Four Singular Points when the Exponent Difference is $\frac{1}{2}$. The

remaining part of this chpater will be devoted to obtaining a classification of differential equations having four singular points with exponent difference equal to $\frac{1}{2}$. We shall use equation (3-1) with $\beta_r = \alpha_r + \frac{1}{2}(r=1,2,3)$ and $\mu_2 = \mu_1 + \frac{1}{2}$, thus

$$(3\text{-}17)\quad u'' + \sum_{r=1}^{3} \frac{\frac{1}{2}-2\alpha_r}{z-a_r}\, u' + \left[\sum_{r=1}^{3} \frac{\alpha_r(\alpha_r+\frac{1}{2})}{(z-a_r)^2} + \frac{Az+B}{\prod_{r=1}^{3}(z-a_r)} \right] u = 0,$$

where $A = \left(\sum_{r=1}^{3} \alpha_r \right)^2 - \sum_{r=1}^{3} \alpha_r^2 - \sum_{r=1}^{3} \alpha_r$.

To verify this expression for A, we recall that the sum and product of the roots of (3-2) will be equal to the negative of the coefficient of μ and the constant term, respectively, thus

$$(3\text{-}18)\qquad 2\mu_1 + \tfrac{1}{2} = -\sum_{r=1}^{3} (2\alpha_r + \tfrac{1}{2}) + 2$$

and

$$(3\text{-}19)\qquad \mu_1(\mu_1+\tfrac{1}{2}) = \sum_{r=1}^{3} \alpha_r(\alpha_r+\tfrac{1}{2}) + A,$$

where we have used the fact that $\beta_r = \alpha_r + \frac{1}{2}$ $(r = 1,2,3)$ and $\mu_2 = \mu_1 + \frac{1}{2}$. From (3-18) we have

$$\mu_1 = -\sum_{r=1}^{3} \alpha_r,$$

which when substituted in (3-19) gives

$$\left(\sum_{r=1}^{3}\alpha_r\right)^2 - \frac{1}{2}\sum_{r=1}^{3}\alpha_r = \sum_{r=1}^{3}\alpha_r^2 + \frac{1}{2}\sum_{r=1}^{3}\alpha_r + A.$$

Therefore,

$$A = \left(\sum_{r=1}^{3}\alpha_r\right)^2 - \sum_{r=1}^{3}\alpha_r^2 - \sum_{r=1}^{3}\alpha_r .$$

Differential equations having four singular points can be classified by the number and nature of the singular points. The coalescence of two singular points having exponent difference $\frac{1}{2}$ produces a regular singularity with an arbitrary exponent difference. For, let $a_2 = a_1$ in (3-17), then

$$(3\text{-}20)\quad u'' + \left[\frac{1-2\alpha_1-2\alpha_2}{z-a_1} + \frac{\frac{1}{2}-2\alpha_3}{z-a_3}\right]u' + \left[\frac{\alpha_1(\alpha_1+\frac{1}{2})+\alpha_2(\alpha_2+\frac{1}{2})}{(z-a_1)^2} + \frac{\alpha_3(\alpha_3+\frac{1}{2})}{(z-a_3)^2} + \frac{Az+B}{(z-a_1)^2(z-a_3)}\right]u = 0.$$

To find the exponent difference at $z = a_1$, we let

$$(3\text{-}21)\qquad u = h_o(z-a_1)^{\lambda} + h_1(z-a_1)^{\lambda+1} + \ldots, \qquad h_o \neq 0.$$

Then

$$u' = h_o\lambda(z-a_1)^{\lambda-1} + h_1(\lambda+1)(z-a_1)^{\lambda} + \ldots,$$

$$u'' = h_o\lambda(\lambda-1)(z-a_1)^{\lambda-2} + h_1(\lambda+1)\lambda(z-a_1)^{\lambda-1} + \ldots.$$

Also note that, expanding about $z = a_1$, we may write

$$\frac{Az+B}{(z-a_1)^2(z-a_3)} = \frac{1}{(z-a_1)^2}\cdot\frac{Az+B}{z-a_3}$$

$$= \frac{1}{(z-a_1)^2}\left[\frac{Aa_1+B}{a_1-a_3} - \frac{a_3A+B}{(a_1-a_3)^2}(z-a_1) + \ldots\right].$$

If (3-21) were then substituted into (3-20) the indicial equation would be found to be

$$(z-a_1)^{\lambda-2}: \quad \lambda(\lambda-1) + (1-2\alpha_1-2\alpha_2)\lambda + \alpha_1(\alpha_1+\tfrac{1}{2}) + \alpha_2(\alpha_2+\tfrac{1}{2}) + \frac{Aa_1+B}{a_1-a_3} = 0,$$

$$\lambda^2 - 2(\alpha_1+\alpha_2)\lambda + \alpha_1(\alpha_1+\tfrac{1}{2}) + \alpha_2(\alpha_2+\tfrac{1}{2}) + \frac{Aa_1+B}{a_1-a_3} = 0.$$

The exponents α and β at the confluent singularity $z = a_1$ are given by the equations

$$\alpha + \beta = 2(\alpha_1+\alpha_2),$$

$$\alpha\beta = \alpha_1(\alpha_1+\tfrac{1}{2}) + \alpha_2(\alpha_2+\tfrac{1}{2}) + \frac{Aa_1+B}{a_1-a_3}.$$

From these equations it can be seen that the exponent difference at the confluent singularity is not $\frac{1}{2}$ but may have any desired value depending on the choice of B.

Further, the coalescence of 3 or more singular points with exponent difference $\frac{1}{2}$ results in an irregular singularity. If $a_1 = a_2 = a_3$, then (3-17) becomes

$$u'' + \frac{\frac{3}{2} - 2\alpha_1 - 2\alpha_2 - 2\alpha_3}{z-a_1} u' + \left[\frac{\alpha_1(\alpha_1+\frac{1}{2}) + \alpha_2(\alpha_2+\frac{1}{2}) + \alpha_3(\alpha_3+\frac{1}{2})}{(z-a_1)^2} + \frac{Az+B}{(z-a_1)^3}\right] u = 0.$$

From our definitions (section 1.1), since

$$(z-a_1)^2 q(z) = (z-a_1)^2 \left[\frac{\alpha_1(\alpha_1+\frac{1}{2})+\alpha_2(\alpha_2+\frac{1}{2})+\alpha_3(\alpha_3+\frac{1}{2})}{(z-a_1)^2} + \frac{Az+B}{(z-a_1)^3} \right]$$

$$= \sum_{r=1}^{3} \alpha_r(\alpha_r+\tfrac{1}{2}) + \frac{Az+B}{z-a_1}$$

is not analytic at $z = a_1$, we have an irregular singular point at $z = a_1$. The nature of an irregular singular point depends on the number of singularities with exponent difference $\frac{1}{2}$ that coalesced to produce it. If three such singularities coalesce, the irregular singularity is said to be of the first species [2, p. 496]; if four such singularities coalesce, it is of the second species.

Now every linear differential equation of the second order with rational coefficients has a definite number of regular and irregular singular points associated with it [2, p. 495]. Since each singularity can be obtained by a confluence of an appropriate number of regular singularities with exponent difference $\frac{1}{2}$, we can consider the equation as derived from one of four standard equations. Let any equation be characterized by a formula [2, p. 497] (a,b,c_1,c_2), where

a = the number of its regular singularities having exponent difference $\frac{1}{2}$,

b = the number of its regular singularities with exponent difference not $\frac{1}{2}$,

c_1 = the number of its irregular singularities of the first species,

c_2 = the number of its irregular singularities of the second species.

Any two equations having the same formula may differ from each other in (a) the location of the singular points, (b) the exponents relative to the regular singular points, and (c) certain arbitrary constants. We shall now take an example of each of the four types of equations that can be obtained from the equation with singular points at $z = a_1, a_2, a_3, \infty$, find the normalized form, and after each present the linear transformations that will carry the general cases over into the particular example. The cases derived from (4,0,0) are:

I. (2,1,0).

II. (0,2,0).

III. (1,0,1).

IV. (0,0,0,1).

I. (2,1,0). The singular points will be at $z = 0,1, \infty, \infty$ for our example, so let $a_1 = 0$, $a_2 = 1$, $a_3 = \infty$ in equation (3-17) to give us

$$(3\text{-}22)\quad u'' + \left[\frac{\frac{1}{2}-2\alpha_1}{z} + \frac{\frac{1}{2}-2\alpha_2}{z-1}\right] u' + \left[\frac{\alpha_1(\alpha_1+\frac{1}{2})}{z^2} + \frac{\alpha_2(\alpha_2+\frac{1}{2})}{(z-1)^2} + \frac{B'}{z(z-1)}\right] u = 0,$$

where A is finite and $B = -a_3B'$.

Now to normalize (3-22) we have from (2-19) that

$$v = e^{-\frac{1}{2}(\frac{1}{2}-2\alpha_1)\ln z - \frac{1}{2}(\frac{1}{2}-2\alpha_2)\ln (z-1)}$$

$$= z^{\alpha_1 - \frac{1}{4}} (z-1)^{\alpha_2 - \frac{1}{4}} .$$

Further

$$v' = (\alpha_1 - \tfrac{1}{4}) z^{\alpha_1 - \frac{5}{4}} (z-1)^{\alpha_2 - \frac{1}{4}} + (\alpha_2 - \tfrac{1}{4}) z^{\alpha_1 - \frac{1}{4}} (z-1)^{\alpha_2 - \frac{5}{4}} .$$

$$v'' = (\alpha_1 - \tfrac{1}{4})(\alpha_1 - \tfrac{5}{4}) z^{\alpha_1 - \frac{9}{4}} (z-1)^{\alpha_2 - \frac{1}{4}}$$
$$+ 2(\alpha_1 - \tfrac{1}{4})(\alpha_2 - \tfrac{1}{4}) z^{\alpha_1 - \frac{5}{4}} (z-1)^{\alpha_2 - \frac{5}{4}}$$
$$+ (\alpha_2 - \tfrac{1}{4})(\alpha_2 - \tfrac{5}{4}) z^{\alpha_1 - \frac{1}{4}} (z-1)^{\alpha_2 - \frac{9}{4}} .$$

With $p(z)$ and $q(z)$ from equation (3-22) and the values of v, v', v'' just found substituted into (2-20) we have

$$z^{\alpha_1 - \frac{1}{4}} (z-1)^{\alpha_2 - \frac{1}{4}} w'' + \Big\{ (\alpha_1 - \tfrac{1}{4})(\alpha_1 - \tfrac{5}{4}) z^{\alpha_1 - \frac{9}{4}} (z-1)^{\alpha_2 - \frac{1}{4}}$$
$$+ 2(\alpha_1 - \tfrac{1}{4})(\alpha_2 - \tfrac{1}{4}) z^{\alpha_1 - \frac{5}{4}} (z-1)^{\alpha_2 - \frac{5}{4}}$$
$$+ (\alpha_2 - \tfrac{1}{4})(\alpha_2 - \tfrac{5}{4}) z^{\alpha_1 - \frac{1}{4}} (z-1)^{\alpha_2 - \frac{9}{4}}$$
$$+ \Big[(\alpha_1 - \tfrac{1}{4}) z^{\alpha_1 - \frac{5}{4}} (z-1)^{\alpha_2 - \frac{1}{4}}$$
$$+ (\alpha_2 - \tfrac{1}{4}) z^{\alpha_1 - \frac{1}{4}} (z-1)^{\alpha_2 - \frac{5}{4}} \Big] \Big[\frac{\frac{1}{2} - 2\alpha_1}{z} + \frac{\frac{1}{2} - 2\alpha_2}{z-1} \Big]$$
$$+ z^{\alpha_1 - \frac{1}{4}} (z-1)^{\alpha_2 - \frac{1}{4}} \Big[\frac{\alpha_1(\alpha_1 + \frac{1}{2})}{z^2} + \frac{\alpha_2(\alpha_2 + \frac{1}{2})}{(z-1)^2} + \frac{B'}{z(z-1)} \Big] \Big\} w = 0.$$

Dividing by $z^{\alpha_1 - \frac{9}{4}} (z-1)^{\alpha_2 - \frac{9}{4}}$,

$$z^2(z-1)^2w'' + \Big[(\alpha_1 - \tfrac{1}{4})(\alpha_1 - \tfrac{5}{4})(z-1)^2 + 2(\alpha_1 - \tfrac{1}{4})(\alpha_2 - \tfrac{1}{4})z(z-1)$$

$$+ (\alpha_2 - \tfrac{1}{4})(\alpha_2 - \tfrac{5}{4})z^2 - 2(\alpha_1 - \tfrac{1}{4})^2(z-1)^2 - 2(\alpha_1 - \tfrac{1}{4})(\alpha_2 - \tfrac{1}{4})z(z-1)$$

$$- 2(\alpha_1 - \tfrac{1}{4})(\alpha_2 - \tfrac{1}{4})z(z-1) - 2(\alpha_2 - \tfrac{1}{4})^2z^2 + \alpha_1(\alpha_1 + \tfrac{1}{2})(z-1)^2$$

$$+ \alpha_2(\alpha_2 + \tfrac{1}{2})z^2 + B'z(z-1)\Big]\ w = 0.$$

$$z^2(z-1)^2w'' + \Big\{\Big[(\alpha_1 - \tfrac{1}{4})(\alpha_1 - \tfrac{5}{4}) + (\alpha_2 - \tfrac{1}{4})(\alpha_2 - \tfrac{5}{4}) - 2(\alpha_1 - \tfrac{1}{4})^2$$

$$- 2(\alpha_1 - \tfrac{1}{4})(\alpha_2 - \tfrac{1}{4}) - 2(\alpha_2 - \tfrac{1}{4})^2 + \alpha_1(\alpha_1 + \tfrac{1}{2}) + \alpha_2(\alpha_2 + \tfrac{1}{2}) + B'\Big]z^2$$

$$+ \Big[-2(\alpha_1 - \tfrac{1}{4})(\alpha_1 - \tfrac{5}{4}) + 4(\alpha_1 - \tfrac{1}{4})^2 + 2(\alpha_1 - \tfrac{1}{4})(\alpha_2 - \tfrac{1}{4})$$

$$- 2\alpha_1(\alpha_1 + \tfrac{1}{2}) - B'\Big]z + (\alpha_1 - \tfrac{1}{4})(\alpha_1 - \tfrac{5}{4}) - 2(\alpha_1 - \tfrac{1}{4})^2 + \alpha_1(\alpha_1 + \tfrac{1}{2})\Big\}\ w = 0.$$

$$z^2(z-1)^2w'' + \Big\{\Big[\tfrac{1}{2}(\alpha_1 + \alpha_2) - 2\alpha_1\alpha_2 + \tfrac{1}{4} + B'\Big]z^2$$

$$+ \Big[-\tfrac{1}{2}(\alpha_1 + \alpha_2) + 2\alpha_1\alpha_2 - \tfrac{1}{4} - B'\Big]z + \tfrac{3}{16}\Big\}\ w = 0.$$

The normalized form is

$$z^2(z-1)^2w'' + \Big[kz(z-1) + \tfrac{3}{16}\Big]\ w = 0,$$

where $k = \tfrac{1}{2}(\alpha_1 + \alpha_2) - 2\alpha_1\alpha_2 + \tfrac{1}{4} + B'$.

The equations with singular points at $\zeta = a,a,b,\infty$ and $\zeta = a,b,\infty,\infty$ can be transformed into the form of (3-22) by taking

$$\zeta = a + \frac{b-a}{z}$$

and

$$\zeta = (b-a)z + a,$$

respectively.

II. (0,2,0). Choose the singular points to be at $z = 0,0,\infty,\infty$, then with $a_1 = a_2 = 0$, $a_3 = \infty$ equation (3-17) becomes

$$(3\text{-}25) \qquad u'' + \frac{1-2\alpha_1-2\alpha_2}{z}\,u' + \left[\frac{\alpha_1(\alpha_1+\frac{1}{2})+\alpha_2(\alpha_2+\frac{1}{2})}{z^2} + \frac{B'}{z^2}\right] u = 0,$$

for A a finite constant and $B = -a_3B'$. To normalize (3-25) use equation (2-19) to obtain

$$v = e^{-\frac{1}{2}(1-2\alpha_1-2\alpha_2)\ln z} = z^{\alpha_1+\alpha_2-\frac{1}{2}}.$$

$$v' = (\alpha_1+\alpha_2-\tfrac{1}{2})z^{\alpha_1+\alpha_2-\frac{3}{2}}.$$

$$v'' = (\alpha_1+\alpha_2-\tfrac{1}{2})(\alpha_1+\alpha_2-\tfrac{3}{2})z^{\alpha_1+\alpha_2-\frac{5}{2}}.$$

Now (2-20) can be written

$$z^{\alpha_1+\alpha_2-\frac{1}{2}}w'' + \Big\{(\alpha_1+\alpha_2-\tfrac{1}{2})(\alpha_1+\alpha_2-\tfrac{3}{2})z^{\alpha_1+\alpha_2-\frac{5}{2}}$$

$$-2(\alpha_1+\alpha_2-\tfrac{1}{2})^2 z^{\alpha_1+\alpha_2-\frac{5}{2}} + \left[\alpha_1(\alpha_1+\tfrac{1}{2})+\alpha_2(\alpha_2+\tfrac{1}{2})+B'\right]z^{\alpha_1+\alpha_2-\frac{5}{2}}\Big\} w=0.$$

Dividing through by $z^{\alpha_1+\alpha_2-\frac{5}{2}}$,

$$z^2w'' + \left[(\alpha_1+\alpha_2-\tfrac{1}{2})(\alpha_1+\alpha_2-\tfrac{3}{2}) - 2(\alpha_1+\alpha_2-\tfrac{1}{2})^2 + \alpha_1(\alpha_1+\tfrac{1}{2}) + \alpha_2(\alpha_2+\tfrac{1}{2}) + B'\right] w = 0.$$

$$z^2w'' + \left[-(\alpha_1+\alpha_2-\tfrac{1}{2})(\alpha_1+\alpha_2+\tfrac{1}{2})+\alpha_1^2 + \alpha_2^2 + \tfrac{1}{2}(\alpha_1+\alpha_2) + B'\right] w = 0.$$

$$z^2w'' + \left[-2\alpha_1\alpha_2 + \tfrac{1}{2}(\alpha_1+\alpha_2) + \frac{1}{4} + B'\right] w = 0.$$

Therefore,

$$z^2w'' + mw = 0,$$

where $m = -2\alpha_1\alpha_2 + \frac{1}{2}(\alpha_1+\alpha_2) + \frac{1}{4} + B'$.

The transformation

$$\zeta = z + a$$

carries the equation with singular points $\zeta = a,a,\infty,\infty$ into an equation of the form (3-25).

III. (1,0,1). Consider the equation with singular points at $z = 0,\infty,\infty,\infty$ which can be obtained from (3-17) by taking $a_1 = 0$, $a_2 = a_3 = \infty$, A a finite constant, $B = a_2a_3B'$:

$$u'' + \frac{\frac{1}{2}-2\alpha_1}{z}\,u' + \left[\frac{\alpha_1(\alpha_1+\frac{1}{2})}{z^2} + \frac{B'}{z}\right] u = 0. \tag{3-26}$$

To normalize (3-26) use (2-19) to find

$$v = z^{\alpha_1-\frac{1}{4}}.$$

Then

$$v' = (\alpha_1 - \tfrac{1}{4}) z^{\alpha_1 - \frac{5}{4}} .$$

$$v'' = (\alpha_1 - \tfrac{1}{4})(\alpha_1 - \tfrac{5}{4}) z^{\alpha_1 - \frac{9}{4}} .$$

Substituting in (2-20),

$$z^{\alpha_1 - \frac{1}{4}} w'' + \Big[(\alpha_1 - \tfrac{1}{4})(\alpha_1 - \tfrac{5}{4}) z^{\alpha_1 - \frac{9}{4}} - 2(\alpha_1 - \tfrac{1}{4})^2 z^{\alpha_1 - \frac{9}{4}} + \alpha_1(\alpha_1 + \tfrac{1}{2}) z^{\alpha_1 - \frac{9}{4}} + B' z^{\alpha_1 - \frac{5}{4}} \Big] w = 0.$$

Dividing by $z^{\alpha_1 - \frac{9}{4}}$,

$$z^2 w'' + \Big[(\alpha_1 - \tfrac{1}{4})(\alpha_1 - \tfrac{5}{4}) - 2(\alpha_1 - \tfrac{1}{4})^2 + \alpha_1(\alpha_1 + \tfrac{1}{2}) + B' z \Big] w = 0.$$

$$z^2 w'' + \left(\frac{3}{16} + B' z \right) w = 0. \qquad (3\text{-}27)$$

This can be further simplified by letting

$$z = \frac{\zeta}{B'} .$$

Then

$$dz = \frac{1}{B'} d\zeta ,$$

$$\frac{dw}{dz} = \frac{dw}{d\zeta} \frac{d\zeta}{dz} = B' \frac{dw}{d\zeta}, \qquad \frac{d^2 w}{dz^2} = (B')^2 \frac{d^2 w}{d\zeta^2} .$$

Thus (3-27) can be written

$$\zeta^2 \frac{d^2 w}{d\zeta^2} + \left(\frac{3}{16} + \zeta \right) w = 0.$$

The transformations

$$\zeta = z + a \qquad \text{and} \qquad \zeta = a + \frac{1}{z}$$

take the equations with singular points at $\zeta = a,\infty,\infty,\infty$ and $\zeta = a,a,a,\infty$, respectively, into an equation of the form (3-26).

IV. (0,0,0,1). In this case the only equation possible is one with all four singular points at $z = \infty$. It will be, from (3-17),

$$\frac{d^2u}{dz^2} + B'u = 0, \tag{3-28}$$

where A is finite and $B = -a_1a_2a_3B'$. If we let

$$z = \frac{\zeta}{\sqrt{B'}},$$

then

$$dz = \frac{1}{\sqrt{B'}}\, d\zeta,$$

$$\frac{du}{dz} = \frac{du}{d\zeta}\frac{d\zeta}{dz} = \sqrt{B'}\,\frac{du}{d\zeta}, \qquad \frac{d^2u}{dz^2} = B'\,\frac{d^2u}{d\zeta^2}.$$

Thus, (3-28) can be written

$$\frac{d^2u}{d\zeta^2} + u = 0.$$

Summary of classification:

Case I. (2,1,0). Normalized form:

$$z^2(z-1)^2w'' + \left[kz(z-1) + \frac{3}{16}\right] w = 0,$$

k is an arbitrary constant.

Case II. (0,2,0). Normalized form:

$$z^2 w'' + mw = 0,$$

m is an arbitrary constant.

Case III. (1,0,1). Normalized form:

$$z^2 w'' + \left(\frac{3}{16} + z\right) w = 0.$$

Case IV. (0,0,0,1). Normalized form:

$$w'' + w = 0.$$

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BIOGRAPHICAL SKETCH

Joyce Coleman Cundiff was born on December 14, 1927 in Erie, Pennsylvania. In May, 1945, she was graduated from Wesleyville High School. She received the degree of Bachelor of Science in mathematics from Westminster College, New Wilmington, Pennsylvania, in June, 1949. After sixteen months with N.A.C.A. in Cleveland, Ohio, as a research computer, she joined the U.S. Air Force for two years. The last year of service was spent at Patrick AFB as a mathematician working with digital computers. Following her discharge she remained at this same job and worked for Radio Corporation of America for two years, taking a six months leave of absence for graduate work at Radcliffe College. In February, 1956, she enrolled in the Graduate School of the University of Florida and received the degree of Master of Science in June, 1957. She held a graduate assistantship from 1956 until 1958. After her marriage to John L. Cundiff in June, 1958, she taught one year at Alabama Polytechnic Institute and one year at the University of Florida. In June, 1960, she began studying full time.

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This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

June 5, 1961

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